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A QUANTUM OCTONION ALGEBRA

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To the memory of Alberto Izquierdo

Abstract. Using the natural irreducible 8-dimensional representation and the two spin representations of the quantum group $U_q(\mathrm{D}_4)$ of D_4, we construct a quantum analogue of the split octonions and study its properties. We prove that the quantum octonion algebra satisfies the q-Principle of Local Triality and has a nondegenerate bilinear form which satisfies a q-version of the composition property. By its construction, the quantum octonion algebra is a nonassociative algebra with a Yang-Baxter operator action coming from the R-matrix of $U_q(\mathrm{D}_4)$. The product in the quantum octonions is a $U_q(\mathrm{D}_4)$-module homomorphism. Using that, we prove identities for the quantum octonions, and as a consequence, obtain at $q = 1$ new "representation theory" proofs for very well-known identities satisfied by the octonions. In the process of constructing the quantum octonions we introduce an algebra which is a q-analogue of the 8-dimensional para-Hurwitz algebra.

Introduction

Using the representation theory of the quantum group $U_q(\mathrm{D}_4)$ of D_4, we construct a quantum analogue $\mathbb{O}_q$ of the split octonions and study its properties.

A unital algebra over a field with a nondegenerate bilinear form $(\,|\,)$ of maximal Witt index which admits composition,

$$(x \cdot y|x \cdot y) = (x|x)(y|y),$$

must be the field, two copies of the field, the split quaternions, or the split octonions. There is a natural q-version of the composition property that the algebra $\mathbb{O}_q$ of quantum octonions is shown to satisfy (see Prop. 4.12 below). We also prove that the quantum octonion algebra $\mathbb{O}_q$ satisfies the "q-Principle of Local Triality" (Prop. 3.12). Inside the quantum octonions are two nonisomorphic 4-dimensional subalgebras, which are q-deformations of the split quaternions. One of them is unital, and both of them give gl_2 when considered as algebras under the commutator product $[x, y] = x \cdot y - y \cdot x$.

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By its construction, $\mathbb{O}_q$ is a nonassociative algebra with a Yang-Baxter operator action coming from the R-matrix of $U_q(\mathrm{D}_4)$. Associative algebras with a Yang-Baxter operator (or r-algebras) arise in Manin's work on noncommutative geometry and include such important examples as Weyl and Clifford algebras, quantum groups, and certain universal enveloping algebras (see for example [B]).

In the process of constructing $\mathbb{O}_q$ we define an algebra $\mathbb{P}_q$, which is a q-analogue of the 8-dimensional para-Hurwitz algebra. The quantum para-Hurwitz algebra $\mathbb{P}_q$ is shown to satisfy certain identities which become familiar properties of the para-Hurwitz algebra at $q = 1$. The para-Hurwitz algebra is a (non-unital) 8-dimensional algebra with a non-degenerate associative bilinear form admitting composition. The algebra $\mathbb{P}_q$ exhibits related properties (see Props. 5.1, 5.3).

While this paper was in preparation, we received a preprint of [Br], which constructs a quantized octonion algebra using the representation theory of $U_q(sl_2)$. Although both Bremner's quantized octonions and our quantum octonions reduce to the octonions at $q = 1$, they are different algebras. The quantized octonion algebra in [Br] is constructed from defining a multiplication on the sum of irreducible $U_q(sl_2)$-modules of dimensions 1 and 7. As a result, it carries a $U_q(sl_2)$-module structure and has a unit element. The quantum octonion algebra constructed in this paper using $U_q(\mathrm{D}_4)$ has a unit element only for the special values $q = 1, -1$.

An advantage to the $U_q(\mathrm{D}_4)$ approach is that it allows properties such as the ones mentioned above to be derived from its representation theory. Using the fact that the product in $\mathbb{O}_q$ is a $U_q(\mathrm{D}_4)$-module homomorphism, we prove identities for $\mathbb{O}_q$ (see Section 4) and as a consequence obtain at $q = 1$ new "representation theory" proofs for very well-known identities satisfied by the octonions that had been established previously by other methods. The $U_q(\mathrm{D}_4)$ approach also affords connections with fixed points of graph automorphisms - although as we show in Section 7, the fixed point subalgebras of $U_q(\mathrm{D}_4)$ are not the quantum groups $U_q(\mathrm{G}_2)$ and $U_q(\mathrm{B}_3)$, because those quantum groups do not have Hopf algebra embeddings into $U_q(\mathrm{D}_4)$. In the final section of this paper we explore connections with quantum Clifford algebras. In particular, we obtain a $U_q(\mathrm{D}_4)$-isomorphism between the quantum Clifford algebra $C_q(8)$ and the endomorphism algebra $\mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$ and discuss its relation to the work of Ding and Frenkel [DF] (compare also [KPS]).

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1. Preliminaries on Quantum Groups

The quantum group $U_q(\mathfrak{g})$. Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$, root system Φ, and simple roots $\Pi = \{\alpha_i \mid i = 1, \ldots, r\} \subset \Phi$ relative to $\mathfrak{h}$. Let $\mathfrak{A} = (a_{\alpha,\beta})_{\alpha,\beta\in\Pi}$ be the corresponding Cartan matrix. Thus,

$$a_{\alpha,\beta} = 2(\beta, \alpha)/(\alpha, \alpha),$$

and there exist relatively prime positive integers,

$$d_\alpha = \frac{(\alpha, \alpha)}{2}, \tag{1.1}$$

so the matrix $(d_\alpha a_{\alpha,\beta})$ is symmetric. These integers are given explicitly by

$$(1.2)\qquad \begin{array}{ll} \mathrm{A}_r, \mathrm{D}_r, \mathrm{E}_r\ (r=6,7,8) & d_{\alpha_i}=1 \text{ for all } 1\le i\le r, \\ \mathrm{B}_r & d_{\alpha_i}=2 \text{ for all } 1\le i\le r-1, \text{ and } d_{\alpha_r}=1, \\ \mathrm{C}_r & d_{\alpha_i}=1 \text{ for all } 1\le i\le r-1, \text{ and } d_{\alpha_r}=2, \\ \mathrm{F}_4 & d_{\alpha_i}=2, i=1,2, \text{ and } d_{\alpha_i}=1, i=3,4, \\ \mathrm{G}_2 & d_{\alpha_1}=1 \text{ and } d_{\alpha_2}=3. \end{array}$$

Let $\mathbf{K}$ be a field of characteristic not 2 or 3. Fix $q\in\mathbf{K}$ such that q is not a root of unity, and such that $q^{\frac{1}{2}}\in\mathbf{K}$. For $m,n,d\in\mathbf{Z}_{>0}$, let

$$(1.3)\qquad [m]_d=\frac{q^{md}-q^{-md}}{q^d-q^{-d}},$$

and set

$$(1.4)\qquad [m]_d!=\prod_{j=1}^{m}[j]_d$$

and $[0]_d!=1$. Let

$$(1.5)\qquad \begin{bmatrix} m\\ n\end{bmatrix}_d=\frac{[m]_d!}{[n]_d![m-n]_d!}.$$

Definition 1.6. The quantum group $U=U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbf{K}$ generated by elements $E_\alpha, F_\alpha, K_\alpha$, and K_α^{-1} (for all $\alpha\in\Pi$) and subject to the relations,

(Q1) $K_\alpha K_\alpha^{-1}=1=K_\alpha^{-1}K_\alpha, \qquad K_\alpha K_\beta=K_\beta K_\alpha,$
(Q2) $K_\alpha E_\beta K_\alpha^{-1}=q^{(\alpha,\beta)}E_\beta,$
(Q3) $K_\alpha F_\beta K_\alpha^{-1}=q^{-(\alpha,\beta)}F_\beta,$
(Q4) $E_\alpha F_\beta-F_\beta E_\alpha=\delta_{\alpha,\beta}\frac{K_\alpha-K_\alpha^{-1}}{q^{d_\alpha}-q^{-d_\alpha}},$
(Q5) $\sum_{s=0}^{1-a_{\alpha,\beta}}(-1)^s\begin{bmatrix}1-a_{\alpha,\beta}\\ s\end{bmatrix}_{d_\alpha}E_\alpha^{1-a_{\alpha,\beta}-s}E_\beta E_\alpha^s=0,$
(Q6) $\sum_{s=0}^{1-a_{\alpha,\beta}}(-1)^s\begin{bmatrix}1-a_{\alpha,\beta}\\ s\end{bmatrix}_{d_\alpha}F_\alpha^{1-a_{\alpha,\beta}-s}F_\beta F_\alpha^s=0.$

Corresponding to each $\lambda=\sum_{\alpha\in\Pi}m_\alpha\alpha$ in the root lattice $\mathbf{Z}\Phi$ there is an element

$$K_\lambda=\prod_{\alpha\in\Pi}K_\alpha^{m_\alpha}$$

in $U_q(\mathfrak{g})$, and $K_\lambda K_\mu=K_{\lambda+\mu}$ for all $\lambda,\mu\in\mathbf{Z}\Phi$. Using relations (Q2),(Q3), we have

$$(1.7)\qquad K_\lambda E_\beta K_\lambda^{-1}=q^{(\lambda,\beta)}E_\beta \qquad\text{and}\qquad K_\lambda F_\beta K_\lambda^{-1}=q^{-(\lambda,\beta)}F_\beta$$

for all $\lambda\in\mathbf{Z}\Phi$ and $\beta\in\Pi$.

The algebra $U_q(\mathfrak{g})$ has a noncocommutative Hopf structure with comultiplication Δ, antipode S, and counit ϵ given by

$$\begin{gathered}\Delta(K_\alpha) = K_\alpha \otimes K_\alpha,\\ \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha,\\ \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha,\\ S(K_\alpha) = K_\alpha^{-1},\\ S(E_\alpha) = -K_\alpha^{-1}E_\alpha, \qquad S(F_\alpha) = -F_\alpha K_\alpha,\\ \epsilon(K_\alpha) = 1, \qquad \epsilon(E_\alpha) = \epsilon(F_\alpha) = 0.\end{gathered} \tag{1.8}$$

The opposite comultiplication Δ^{op} has the property that

$$\text{if } \Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then} \quad \Delta^{\mathrm{op}}(a) = \sum_a a_{(2)} \otimes a_{(1)}.$$

(We are adopting the commonly used Sweedler notation for components of the comultiplication.) The algebra U with the same multiplication, same identity, and same counit, but with comultiplication given by Δ^{op} and with S^{-1} as its antipode, is also a Hopf algebra. The comultiplication Δ^{op} can be regarded as the composition $\tau \circ \Delta$, where $\tau : U \otimes U \to U \otimes U$ is given by $\tau(a \otimes b) = b \otimes a$. The map $\psi : U \to U_q(\mathfrak{g})$ with $E_\alpha \mapsto F_\alpha$, $F_\alpha \mapsto E_\alpha$, $K_\alpha \mapsto K_\alpha^{-1}$ is an algebra automorphism, and a coalgebra anti-automorphism, so U is isomorphic to $U_q(\mathfrak{g})$ as a Hopf algebra (see [CP, p. 211]).

The defining property of the antipode S in any Hopf algebra is that it is an inverse to the identity map id_U with respect to the convolution product, so that for all $a \in U$,

$$\sum_a S(a_{(1)})a_{(2)} = \epsilon(a)\mathrm{id}_U = \sum_a a_{(1)}S(a_{(2)}). \tag{1.9}$$

Representations. For any Hopf algebra U the comultiplication operator allows us to define a U-module structure on the tensor product $V \otimes W$ of two U-modules V and W, where

$$a(v \otimes w) = \sum_a a_{(1)}v \otimes a_{(2)}w.$$

The counit ϵ affords a representation of U on the one-dimensional module given by $a1 = \epsilon(a)1$. If V is a finite-dimensional U-module, then so is its dual space V^*, where the U-action is given by the antipode mapping,

$$(af)(v) = f(S(a)v), \tag{1.10}$$

for all $a \in U$, $f \in V^*$, and $v \in V$.

A finite-dimensional module M for $U = U_q(\mathfrak{g})$ is the sum of weight spaces, $M = \bigoplus_{\nu,\chi} M_{\nu,\chi}$, where $M_{\nu,\chi} = \{x \in M \mid K_\alpha x = \chi(\alpha)q^{(\nu,\alpha)}x \text{ for all } \alpha \in \Pi\}$, and $\chi : \mathbf{Z}\Phi \to \{\pm 1\}$ is a group homomorphism. All the modules considered in this paper will be of type I, that is, $\chi(\alpha) = 1$ for all $\alpha \in \Pi$, so we will drop χ from the notation and from our considerations.

Each finite-dimensional irreducible U-module M has a highest weight λ, which is a dominant integral weight for $\mathfrak{g}$ relative to Π, and a unique (up to scalar multiple) maximal vector v^+ so that $E_\alpha v^+ = 0$ and $K_\alpha v^+ = q^{(\lambda,\alpha)}v^+$ for all $\alpha \in \Pi$. In particular, the trivial U-module $\mathbf{K}1$ has highest weight 0. We denote by $L(\lambda)$ the irreducible U-module with highest weight λ (and $\chi \equiv 1$).

When V is a finite-dimensional U-module, we can suppose $\{x_{\nu,i}\}$ is a basis of V of weight vectors with $i = 1, \dots, \dim V_\nu$. Let $\{x^*_{\nu,i}\}$ be the dual basis in V^*, so that $x^*_{\mu,i}(x_{\nu,j}) = \delta_{\mu,\nu}\delta_{i,j}$. Then

$$(1.11)\qquad \begin{aligned}(K_\alpha x^*_{\mu,i})(x_{\nu,j}) &= x^*_{\mu,i}(S(K_\alpha)x_{\nu,j}) = x^*_{\mu,i}(K_\alpha^{-1}x_{\nu,j})\\ &= q^{-(\nu,\alpha)}x^*_{\mu,i}(x_{\nu,j}) = q^{-(\nu,\alpha)}\delta_{\mu,\nu}\delta_{i,j} = q^{-(\mu,\alpha)}\delta_{\mu,\nu}\delta_{i,j},\end{aligned}$$

from which we see that $x^*_{\mu,i}$ has weight $-\mu$.

Suppose M and N are U-modules for a quantum group U. Define a U-module structure on $\mathrm{Hom}_{\mathbf{K}}(M,N)$ by

$$(1.12)\qquad (a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m).$$

(This just means that the natural map $N \otimes M^* \to \mathrm{Hom}_{\mathbf{K}}(M,N)$ is a U-module homomorphism.)

Note that if $\gamma \in \mathrm{Hom}_U(M,N)$, then

$$(a\gamma)(m) = \sum_a a_{(1)}\gamma(S(a_{(2)})m) = \sum_a a_{(1)}S(a_{(2)})\gamma(m) = \epsilon(a)\gamma(m),$$

so that $a\gamma = \epsilon(a)\gamma$. That says

$$\mathrm{Hom}_U(M,N) \subseteq \mathrm{Hom}_{\mathbf{K}}(M,N)^U = \{\gamma \in \mathrm{Hom}_{\mathbf{K}}(M,N) \mid a\gamma = \epsilon(a)\gamma\}.$$

Conversely, if γ belongs to the invariants $\mathrm{Hom}_{\mathbf{K}}(M,N)^U$, then we have

$$\begin{aligned}0 = \epsilon(K_\alpha)\gamma &= \gamma = K_\alpha \circ \gamma \circ K_\alpha^{-1}, \quad \text{and hence,}\\ 0 &= K_\alpha \circ \gamma - \gamma \circ K_\alpha;\\ 0 = \epsilon(E_\alpha)\gamma &= E_\alpha\gamma = E_\alpha \circ \gamma - K_\alpha \circ \gamma \circ K_\alpha^{-1} \circ E_\alpha, \quad \text{which implies}\\ 0 &= E_\alpha \circ \gamma - \gamma \circ E_\alpha;\\ 0 = \epsilon(F_\alpha)\gamma &= F_\alpha\gamma = F_\alpha \circ \gamma \circ K_\alpha - \gamma \circ F_\alpha \circ K_\alpha, \quad \text{which implies}\\ 0 &= F_\alpha \circ \gamma - \gamma \circ F_\alpha\end{aligned}$$

because of the relations in (1.8). Since these elements generate $U_q(\mathfrak{g})$, we see that $\gamma \in \mathrm{Hom}_{\mathbf{K}}(M,N)^U$ implies $\gamma \in \mathrm{Hom}_U(M,N)$, so the two spaces are equal:

$$(1.13)\qquad \mathrm{Hom}_U(M,N) = \mathrm{Hom}_{\mathbf{K}}(M,N)^U.$$

Now if $M = N = L(\lambda)$, a finite-dimensional irreducible U-module of highest weight λ, then any $f \in \mathrm{Hom}_U(M,M)$ must map the highest weight vector v^+ to a multiple of itself. Since v^+ generates M as a U-module, it follows that f is a scalar multiple of the identity. Thus, $\mathrm{Hom}_U(M,M) = \mathbf{K}\mathrm{id}_M$. So the space of invariants $\mathrm{Hom}_{\mathbf{K}}(M,M)^U$, or equivalently $(M \otimes M^*)^U$, is one-dimensional.

R-matrices. The quantum group $U = U_q(\mathfrak{g})$ has a triangular decomposition $U = U^-U^0U^+$, where U^+ (resp. U^-) is the subalgebra generated by the E_α (resp. F_α), and U^0 is the subalgebra generated by the $K_\alpha^{\pm 1}$, for all $\alpha \in \Pi$. Then $U^+ = \sum_\mu U^+_\mu$, where $U^+_\mu = \{a \in U^+ \mid K_\alpha a K_\alpha^{-1} = q^{(\mu,\alpha)}a\}$. It is easy to see, using the automorphism ψ above, that U^- has the decomposition $U^- = \sum_\mu U^-_{-\mu}$, and $U^+_\mu \neq (0)$ if and only if $U^-_{-\mu} \neq (0)$. There is a nondegenerate bilinear form $(\ ,\ )$ on

U (see [Jn, Lemma 6.16, Prop. 6.21]) which satisfies

(1.14) (a) $(b,a) = q^{(2\rho,\mu-\nu)}(a,b)$ for all $a \in U^-_{-\nu}U^0U^+_\mu$ and $b \in U^-_{-\mu}U^0U^+_\nu$, where ρ is the half-sum of the positive roots of $\mathfrak{g}$.
(b) $(F_\alpha, E_\beta) = -\delta_{\alpha,\beta}(q_\alpha - q_\alpha^{-1})^{-1}$.

Choose a basis $a_1^\mu, a_2^\mu, \dots, a_{r(\mu)}^\mu$ of U_μ^+ and a corresponding dual basis $b_1^\mu, b_2^\mu, \dots, b_{r(\mu)}^\mu$ of $U^-_{-\mu}$ so that $(b_i^\mu, a_j^\mu) = \delta_{i,j}$. Set

$$\Theta_\mu = \sum_{i=1}^{r(\mu)} b_i^\mu \otimes a_i^\mu.$$

In particular, $\Theta_\alpha = -(q_\alpha - q_\alpha^{-1})F_\alpha \otimes E_\alpha$ for all $\alpha \in \Pi$.

Suppose $\pi : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ and $\pi' : U_q(\mathfrak{g}) \to \mathrm{End}(V')$ are two finite-dimensional $U_q(\mathfrak{g})$-modules. Since the set of weights of V is finite, there are only finitely many $\mu \in \mathbf{Z}\Phi$ which are differences of weights of V. Therefore, all but finitely many Θ_μ act as zero on V and V', and $\Theta_{V,V'} = (\pi \otimes \pi')(\Theta)$ for $\Theta = \sum_\mu \Theta_\mu$ is a well-defined mapping on $V \otimes V'$. By suitably ordering the basis of $V \otimes V'$, we see that each Θ_μ for $\mu \neq 0$ acting on $V \otimes V'$ has a strictly upper triangular matrix relative to that basis. Since $\Theta_0 = 1 \otimes 1$, the transformation $\Theta_{V,V'}$ is unipotent.

Consider the map $f = f_{V,V'}$, which is defined by $f_{V,V'} : V \otimes V' \to V \otimes V'$, $f(v \otimes w) = q^{-(\lambda,\mu)} v \otimes w$ for all $v \in V_\lambda$ and $w \in V'_\mu$.

Proposition 1.15 (Compare [Jn, Thm. 7.3]). *Let V and V' be finite-dimensional $U_q(\mathfrak{g})$-modules, and let $\sigma = \sigma_{V',V}$ be defined by*

$$\sigma : V' \otimes V \to V \otimes V',$$
$$w \otimes v \mapsto v \otimes w.$$

Then the mapping $\check{R}_{V',V} : V' \otimes V \to V \otimes V'$ given by $\check{R}_{V',V} = \Theta \circ f \circ \sigma$ is a $U_q(\mathfrak{g})$-module isomorphism.

There is an alternate expression for Θ_μ which is more explicit and can be best understood using the braid group action. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators T_α, $\alpha \in \Pi$, and defining relations

$$(T_\alpha T_\beta)^{m_{\alpha,\beta}} = (T_\beta T_\alpha)^{m_{\alpha,\beta}},$$

where $m_{\alpha,\beta} = 2,3,4,6$ if $a_{\alpha,\beta}a_{\beta,\alpha} = 0,1,2,3$ respectively. It acts by algebra automorphisms on $U_q(\mathfrak{g})$ according to the following rules (see [Jn, Sec. 8.14]):

$$T_\alpha(K_\beta) = K_{s_\alpha\beta}, \qquad T_\alpha(E_\alpha) = -F_\alpha K_\alpha, \qquad T_\alpha(F_\alpha) = -K_\alpha^{-1}E_\alpha,$$

(1.16)
$$T_\alpha(E_\beta) = \sum_{t=0}^{-a_{\alpha,\beta}} (-1)^t q_\alpha^{-t} (E_\alpha)^{(-a_{\alpha,\beta}-t)} E_\beta (E_\alpha)^{(t)}, \quad \alpha \neq \beta,$$
$$T_\alpha(F_\beta) = \sum_{t=0}^{-a_{\alpha,\beta}} (-1)^t q_\alpha^{t} (F_\alpha)^{(t)} F_\beta (F_\alpha)^{(-a_{\alpha,\beta}-t)}, \quad \alpha \neq \beta,$$

where s_α is the simple reflection corresponding to α in the Weyl group W of $\mathfrak{g}$, and

$$(E_\alpha)^{(n)} = \frac{E_\alpha^n}{[n]_{d_\alpha}!}, \qquad (F_\alpha)^{(n)} = \frac{F_\alpha^n}{[n]_{d_\alpha}!}$$

(the factorials are as defined in (1.4)).

Let $s_i = s_{\alpha_i}$ be the reflection in the hyperplane perpendicular to the simple root α_i, and fix a reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_m}$ of the longest element of W. Every positive root occurs precisely once in the sequence

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \ \ldots, \ \beta_m = s_{i_1} s_{i_2} \cdots s_{i_{m-1}}(\alpha_{i_m}). \tag{1.17}$$

The elements

$$\begin{aligned} E_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}}(E_{\alpha_{i_t}}), \\ F_{\beta_t} &= T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_{t-1}}}(F_{\alpha_{i_t}}), \end{aligned} \tag{1.18}$$

for $t = 1, \ldots, m$, are called the positive (resp. negative) root vectors of $U_q(\mathfrak{g})$. For any m-tuple $\underline{\ell} = (\ell_1, \ldots, \ell_m)$ of nonnegative integers let

$$\begin{aligned} E^{\underline{\ell}} &= E_{\beta_m}^{\ell_m} E_{\beta_{m-1}}^{\ell_{m-1}} \cdots E_{\beta_1}^{\ell_1} \\ F^{\underline{\ell}} &= F_{\beta_m}^{\ell_m} F_{\beta_{m-1}}^{\ell_{m-1}} \cdots F_{\beta_1}^{\ell_1}. \end{aligned}$$

Then the elements $E^{\underline{\ell}}$ (resp. $F^{\underline{\ell}}$) determine a basis of U^+ (resp. U^-). (See for example, [Jn, Thm. 8.24].)

Consider for $t = 1, \ldots, m$ the sum

$$\Theta^{[t]} = \sum_{j \geq 0} (-1)^j q_\alpha^{-j(j-1)/2} \frac{(q_\alpha - q_\alpha^{-1})^j}{[j]_{d_\alpha}!} F_{\beta_t}^j \otimes E_{\beta_t}^j,$$

where $\alpha = \alpha_{i_t}$, and E_{β_t} and F_{β_t} are as in (1.18). This lies in the direct product of all the $U_{-\mu}^- \otimes U_\mu^+$. Then (as discussed in [Jn, Sec. 8.30]), Θ_μ is the $(U_\mu^- \otimes U_\mu^+)$-component of the product

$$\Theta^{[m]} \Theta^{[m-1]} \cdots \Theta^{[2]} \Theta^{[1]}.$$

That component involves only finitely many summands.

2. 8-dimensional Representations of $U_q(\mathrm{D}_4)$

In this section we specialize to the case that $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra of type D_4. We can identify $\mathfrak{g}$ with the special orthogonal Lie algebra $so(8)$. The set of roots of $\mathfrak{g}$ is given by $\{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i \neq j \leq 4\}$, where $\epsilon_i, i = 1, \ldots, 4$, is an orthonormal basis of $\mathbf{R}^4$, and the simple roots may be taken to be $\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3,\ \alpha_3 = \epsilon_3 - \epsilon_4,\ \alpha_4 = \epsilon_3 + \epsilon_4$. The corresponding fundamental weights have the following expressions:

$$\begin{aligned} \omega_1 &= \alpha_1 + \alpha_2 + (1/2)\alpha_3 + (1/2)\alpha_4 = \epsilon_1, \\ \omega_2 &= \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \epsilon_1 + \epsilon_2, \\ \omega_3 &= (1/2)\alpha_1 + \alpha_2 + \alpha_3 + (1/2)\alpha_4 = (1/2)\Big(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4\Big), \\ \omega_4 &= (1/2)\alpha_1 + \alpha_2 + (1/2)\alpha_3 + \alpha_4 = (1/2)\Big(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4\Big), \end{aligned} \tag{2.1}$$

and they satisfy $(\omega_i, \alpha_j) = \delta_{i,j}$ for all $i, j = 1, \ldots, 4$.

The Lie algebra $\mathfrak{g}$ has three 8-dimensional representations—its natural representation and two spin representations with highest weights ω_1, ω_3, and ω_4 respectively, and so does its quantum counterpart $U_q(\mathrm{D}_4)$. (*Note we are writing* $U_q(\mathrm{D}_4)$ *rather than* $U_q(\mathfrak{g})$ *to emphasize the role that* D_4 *is playing here.*) In particular, for the natural representation V of $U_q(\mathrm{D}_4)$, there is a basis $\{x_\mu \mid \mu = \pm\epsilon_i, i = 1, \dots, 4\}$ (compare [Jn, Chap. 5]) such that

$$\begin{aligned} K_\alpha x_\mu &= q^{(\mu,\alpha)} x_\mu, \\ E_\alpha x_\mu &= \begin{cases} 0 & \text{if} \quad (\mu,\alpha) \neq -1, \\ x_{\mu+\alpha} & \text{if} \quad (\mu,\alpha) = -1, \end{cases} \\ F_\alpha x_\mu &= \begin{cases} 0 & \text{if} \quad (\mu,\alpha) \neq 1, \\ x_{\mu-\alpha} & \text{if} \quad (\mu,\alpha) = 1. \end{cases} \end{aligned} \tag{2.2}$$

For D_4, the group of graph automorphisms of its Dynkin diagram can be identified with the symmetric group S_3 of permutations on $\{1, 2, 3, 4\}$ fixing 2. Thus, $\phi \in S_3$ permutes the simple roots $\phi(\alpha_i) = \alpha_{\phi i}$ and fixes α_2. Each graph automorphism ϕ induces an automorphism, which we also denote by ϕ, on $U_q(\mathrm{D}_4)$ defined by

$$E_{\alpha_i} \mapsto E_{\alpha_{\phi i}} \qquad F_{\alpha_i} \mapsto F_{\alpha_{\phi i}} \qquad K_{\alpha_i} \mapsto K_{\alpha_{\phi i}}.$$

Corresponding to ϕ there is a new representation $\pi_\phi : U_q(\mathrm{D}_4) \to \mathrm{End}(V)$ of $U_q(\mathrm{D}_4)$ on its natural representation V such that

$$\pi_\phi(a)x = \phi(a)x$$

for all $a \in U_q(\mathrm{D}_4)$ and $x \in V$. We denote the $U_q(\mathrm{D}_4)$-module with this action by V_ϕ. Then we have

$$\begin{aligned} \pi_\phi(K_\alpha)x_\mu &= K_{\phi\alpha}x_\mu = q^{(\mu,\phi\alpha)}x_\mu = q^{(\phi^{-1}\mu,\alpha)}x_\mu, \\ \pi_\phi(E_\alpha)x_\mu &= E_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) \neq -1, \\ x_{\mu+\phi\alpha} & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) = -1, \end{cases} \\ \pi_\phi(F_\alpha)x_\mu &= F_{\phi\alpha}x_\mu = \begin{cases} 0 & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) \neq 1, \\ x_{\mu-\phi\alpha} & \text{if} \quad (\phi^{-1}\mu,\alpha) = (\mu,\phi\alpha) = 1. \end{cases} \end{aligned} \tag{2.3}$$

From these formulas the following is apparent:

Proposition 2.4. *V_ϕ is an irreducible $U_q(\mathrm{D}_4)$-module with highest weight $\phi^{-1}\omega_1$. In particular, we have the following highest weights:*

(i) ω_1 *for* $\phi = \mathrm{id}$ *or* $(3\ 4)$,
(ii) ω_3 *for* $\phi = (1\ 3)$ *or* $(1\ 4\ 3)$,
(iii) ω_4 *for* $\phi = (1\ 4)$ *or* $(1\ 3\ 4)$.

The operators E_α, F_α map elements in the basis $\{x_\mu\}$ to other basis elements or 0, and each vector x_μ is a weight vector of weight $\phi^{-1}\mu$. For the natural representation V, the basis elements are $x_{\pm\epsilon_i}, i = 1, \dots, 4$, which we abbreviate as $x_{\pm i}$. We can view the module V_ϕ as displayed below, where the index i on an arrow indicates that F_{α_i} maps the higher vector down to the lower vector and E_{α_i} maps the lower vector up to the higher one. The action of all F_{α_j}'s and E_{α_j}'s not shown is 0.

$$V_\phi$$

$$\begin{array}{ccccc}
 & & x_1 \circ & & \\
 & & \downarrow \phi^{-1} 1 & & \\
 & & x_2 \circ & & \\
 & & \downarrow 2 & & \\
 & & x_3 \circ & & \\
 & \phi^{-1}3 \swarrow & & \searrow \phi^{-1}4 & \\
x_4 \circ & & & & \circ\, x_{-4} \\
 & \phi^{-1}4 \searrow & & \swarrow \phi^{-1}3 & \\
 & & x_{-3} \circ & & \\
 & & \downarrow 2 & & \\
 & & x_{-2} \circ & & \\
 & & \downarrow \phi^{-1} 1 & & \\
 & & x_{-1} \circ & &
\end{array} \tag{2.5}$$

In particular, taking the identity, $\theta = (1\ 4\ 3)$, and $\theta^2 = (1\ 3\ 4)$, we have the following pictures for the natural representation V, and the spin representations V_- and V_+:

(2.6)

$$\begin{array}{ccc}
V & V_- & V_+ \\
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 1 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
3 \swarrow & & \searrow 4 \\
x_4 \circ & & \circ\, x_{-4} \\
4 \searrow & & \swarrow 3 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 1 & \\
 & x_{-1}\circ &
\end{array} &
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 3 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
4 \swarrow & & \searrow 1 \\
x_4 \circ & & \circ\, x_{-4} \\
1 \searrow & & \swarrow 4 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 3 & \\
 & x_{-1}\circ &
\end{array} &
\begin{array}{ccc}
 & x_1\circ & \\
 & \downarrow 4 & \\
 & x_2\circ & \\
 & \downarrow 2 & \\
 & x_3\circ & \\
1 \swarrow & & \searrow 3 \\
x_4 \circ & & \circ\, x_{-4} \\
3 \searrow & & \swarrow 1 \\
 & x_{-3}\circ & \\
 & \downarrow 2 & \\
 & x_{-2}\circ & \\
 & \downarrow 4 & \\
 & x_{-1}\circ &
\end{array}
\end{array}$$

There is a unique (up to scalar multiple) $U_q(\mathrm{D}_4)$-module homomorphism $* : V_- \otimes V_+ \to V$, which we can compute directly using the expressions for the co-multiplication in (1.8). We display the result of this computation in Table 2.7. To make the table more symmetric, we scale the mapping by $q^{\frac{3}{2}}$.

Table 2.7

$*$	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	$q^{-\frac{3}{2}}x_1$	0	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_2$	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_3$	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	$-q^{\frac{3}{2}}x_1$	0	0	0	$-q^{-\frac{1}{2}}x_{-4}$	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-4}	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	$q^{\frac{1}{2}}x_4$	0	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	$q^{\frac{1}{2}}x_{-3}$	0	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-2}$	0	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	0	$-q^{\frac{3}{2}}x_{-1}$	0	0	0

It can be seen readily from (1.8) that the relations

$$(\phi \otimes \phi)\Delta(a) = \Delta(\phi(a)), \quad \phi(S(a)) = S(\phi(a)) \tag{2.8}$$

hold for all $a \in U_q(\mathrm{D}_4)$ and all graph automorphisms ϕ. Therefore, by identifying V_- with V_θ and V_+ with V_{θ^2} we obtain from

$$a(x * y) = \sum_a \theta(a_{(1)})x * \theta^2(a_{(2)})y$$

that the following relations hold:

$$\begin{aligned} \pi_\theta(a)(x * y) &= \sum_a \theta^2(a_{(1)})x * a_{(2)}y, \\ \pi_{\theta^2}(a)(x * y) &= \sum_a a_{(1)}x * \theta(a_{(2)})y. \end{aligned} \tag{2.9}$$

They allow us to conclude

Proposition 2.10. *The product $* : V_- \otimes V_+ \to V$ displayed in Table 2.7 is a $U_q(\mathrm{D}_4)$-module homomorphism, and it gives $U_q(\mathrm{D}_4)$-module homomorphisms*

$$* : V_+ \otimes V \to V_-, \quad * : V \otimes V_- \to V_+.$$

Suppose now that $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Specializing (2.5) with ϕ equal to these permutations gives

$$\begin{array}{ccc} & V_\zeta & \\ & x_1\circ & \\ & \downarrow 3 & \\ & x_2\circ & \\ & \downarrow 2 & \\ & x_3\circ & \\ 1\swarrow & & \searrow 4 \\ x_4\circ & & \circ x_{-4} \\ 4\searrow & & \swarrow 1 \\ & x_{-3}\circ & \\ & \downarrow 2 & \\ & x_{-2}\circ & \\ & \downarrow 3 & \\ & x_{-1}\circ & \end{array} \qquad \begin{array}{ccc} & V_\eta & \\ & x_1\circ & \\ & \downarrow 4 & \\ & x_2\circ & \\ & \downarrow 2 & \\ & x_3\circ & \\ 3\swarrow & & \searrow 1 \\ x_4\circ & & \circ x_{-4} \\ 1\searrow & & \swarrow 3 \\ & x_{-3}\circ & \\ & \downarrow 2 & \\ & x_{-2}\circ & \\ & \downarrow 4 & \\ & x_{-1}\circ & \end{array} \tag{2.11}$$

As these diagrams indicate, the transformation specified by

$$\begin{aligned} \jmath : V &\to V, \\ x_i &\mapsto -x_i, \qquad i \neq \pm 4, \\ x_4 &\mapsto -x_{-4}, \\ x_{-4} &\mapsto -x_4 \end{aligned} \tag{2.12}$$

determines explicit $U_q(\mathrm{D}_4)$-isomorphisms,

$$\jmath : V_- \to V_\zeta \qquad \jmath : V_+ \to V_\eta. \tag{2.13}$$

This together with Proposition 2.10 enables us to conclude the following:

Proposition 2.14. *The map* $\cdot : V_\zeta \otimes V_\eta \to V$ *given by* $x \otimes y \mapsto x \cdot y = \jmath(x) * \jmath(y)$ *is a* $U_q(\mathrm{D}_4)$*-module homomorphism.*

Eigenvalues of $\check{R}_{V,V}$. Suppose now that $V = L(\omega_1)$, the natural representation of $U_q(\mathrm{D}_4)$. The module $V^{\otimes 2}$ is completely reducible: $V^{\otimes 2} = L(2\omega_1) \oplus L(\omega_2) \oplus L(0)$. By determining the eigenvalues of $\check{R}_{V,V}$ on the corresponding maximal vectors, we obtain the eigenvalues of $\check{R}_{V,V}$ on the summands.

First, on the maximal vector $x_1 \otimes x_1$ of the summand $L(2\omega_1)$, $\check{R}_{V,V}(x_1 \otimes x_1) = q^{-(\epsilon_1,\epsilon_1)} x_1 \otimes x_1 = q^{-1}(x_1 \otimes x_1)$. The vector $x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2$ is a maximal vector for $L(\omega_2)$. Then

$$\begin{aligned} \check{R}_{V,V}(x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2) &= q^{-(\epsilon_1,\epsilon_2)} \Theta(x_1 \otimes x_2 - q^{-1} x_2 \otimes x_1) \\ &= x_1 \otimes x_2 - q^{-1} x_2 \otimes x_1 - (q - q^{-1}) x_2 \otimes x_1 \\ &= x_1 \otimes x_2 - q x_2 \otimes x_1 = -q(x_2 \otimes x_1 - q^{-1} x_1 \otimes x_2), \end{aligned}$$

so that $\check{R}_{V,V}$ acts as $-q$ on $L(\omega_2)$. We will argue in the remark following Proposition 2.19 below that $\check{R}_{V,V}$ acts on $L(0)$ as q^7. Thus, the eigenvalues of $\check{R}_{V,V}$ are given by

$$\begin{aligned} q^{-1}, &\qquad L(2\omega_1), \\ -q, &\qquad L(\omega_2), \\ q^7, &\qquad L(0), \end{aligned} \tag{2.15}$$

(compare [D], [R]). Setting $\mathcal{S}^2(V) = L(2\omega_1) \oplus L(0)$, we see that $\check{R}_{V,V} + q\mathrm{id}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$, the *quantum symmetric space*. There are analogous decompositions for the two spin representations of $U_q(\mathrm{D}_4)$ obtained by replacing ω_1 by ω_3 and ω_4.

For each graph automorphism ϕ, we consider $\check{R} = \check{R}_{V_\phi,V_\phi}$ on $V_\phi \otimes V_\phi$. Note the bilinear form (,) on $U_q(\mathrm{D}_4)$ in [Jn, (6.12)] is $(\phi \otimes \phi)$-invariant, so

$$\Theta_{\phi(\mu)} = \sum_i \phi(b_i^\mu) \otimes \phi(a_i^\mu) = (\phi \otimes \phi)(\Theta_\mu).$$

Now

$$\begin{aligned} \check{R}(v \otimes w) &= (\Theta \circ f \circ \sigma)(v \otimes w) = q^{-(\phi^{-1}\lambda, \phi^{-1}\nu)} \Theta(w \otimes v) \\ &= q^{-(\lambda,\nu)} \Theta(w \otimes v) = \check{R}_{V,V}(v \otimes w), \end{aligned} \tag{2.16}$$

whenever w is in V_λ and $v \in V_\nu$. Consequently, the actions of the R-matrices agree. It follows that

$$\mathcal{S}^2(V) = \mathcal{S}^2(V_\phi) \tag{2.17}$$

for all graph automorphisms ϕ.

A "test" element of $\mathcal{S}^2(V)$. In the next two sections we will compute various $U_q(\mathrm{D}_4)$-module homomorphisms and prove that they are equal by evaluating them on a particular element of $\mathcal{S}^2(V)$. A good "test" element for these calculations is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$, but first we need to demonstrate that it does in fact belong to $\mathcal{S}^2(V)$. Since $\check{R}_{V,V} + q\mathrm{id}_V$ maps $V^{\otimes 2}$ onto $\mathcal{S}^2(V)$ for $V = L(\omega_1)$, the natural representation, it suffices to show that the image of $x_1 \otimes x_{-1}$ under $\check{R}_{V,V} + q\mathrm{id}_V$ is $x_1 \otimes x_{-1} + x_{-1} \otimes x_1$. Using the fact that $E_\gamma x_1 = 0$ for all positive roots γ, we obtain

$$\begin{aligned}\check{R}_{V,V}(x_1 \otimes x_{-1}) &= (\Theta \circ f \circ \sigma)(x_1 \otimes x_{-1}) \\ &= q\Theta(x_{-1} \otimes x_1) = q(x_{-1} \otimes x_1).\end{aligned}$$

Consequently,

$$(\check{R}_{V,V} + q\mathrm{id}_V)(x_1 \otimes x_{-1}) = qx_{-1} \otimes x_1 + qx_1 \otimes x_{-1},$$

which implies the desired conclusion

$$x_{-1} \otimes x_1 + x_1 \otimes x_{-1} \in \mathcal{S}^2(V).$$

Bilinear forms. For the natural representation V of $U = U_q(\mathrm{D}_4)$, the dual space V^* has weights $\pm\epsilon_i$, $i = 1, \ldots, 4$, by (1.11). The vector x_{-1}^* is a maximal vector since it corresponds to the unique dominant weight, ϵ_1, and V^* is isomorphic to V. We have seen in (1.13) that the space of invariants $(V \otimes V^*)^U$ is one-dimensional. Since $V \cong V^*$, we have that the space of invariants $(V \otimes V)^U$ is one-dimensional. Thus, in $V \otimes V$ there is a unique copy of the trivial module, and hence a unique (up to scalar multiple) $U_q(\mathrm{D}_4)$-module homomorphism

$$(\,|\,) V \otimes V \to \mathbf{K}.$$

The condition of $(\,|\,)$ being a $U_q(\mathrm{D}_4)$-module homomorphism is equivalent to

$$(ax|y) = (x|S(a)y) \tag{2.18}$$

for all $a \in U$ and $x, y \in V$ (see [Jn, p.122]). There is an analogous bilinear form giving the unique (up to multiples) $U_q(\mathrm{D}_4)$-module homomorphism, $V_\phi \otimes V_\phi \to \mathbf{K}$. All these bilinear forms can be regarded as $U_q(\mathrm{D}_4)$-module maps from the symmetric tensors $\mathcal{S}^2(V_\phi) \to \mathbf{K}$, since the trivial module is an irreducible summand of $\mathcal{S}^2(V_\phi)$ in each case.

Now for $a \in U_q(\mathrm{D}_4)$,

$$
\begin{aligned}
\sum_a (\pi_\phi(a_{(1)})x|\pi_\phi(a_{(2)})y) &= \sum_a (\phi(a_{(1)})x|\phi(a_{(2)})y) \\
&= \sum_a (x|S(\phi(a_{(1)}))\phi(a_{(2)})y) \\
&= \sum_a (x|\phi(S(a_{(1)})a_{(2)})y \\
&= (x|\phi\Big(\sum_a S(a_{(1)})a_{(2)}\Big))y \\
&= (x|\phi(\epsilon(a)1)y) = \epsilon(a)(x|y).
\end{aligned} \tag{2.19}
$$

(We have used in the third line the fact that ϕ and S commute (2.8).) Hence, by the calculation in (2.19), and by direct computation of $(\,|\,)$ on $V \otimes V$, we have

Proposition 2.20. *The $U_q(\mathrm{D}_4)$-module homomorphism $(\,|\,) : V \otimes V \to \mathbf{K}$ determines a $U_q(\mathrm{D}_4)$-module homomorphism $(\,|\,) : V_\phi \otimes V_\phi \to \mathbf{K}$ for any graph automorphism ϕ. It is given by*

$$
(x_i|x_j) = \begin{cases} 0 & \text{if} \quad j \neq -i, \\ (-1)^{i+1}\dfrac{q^{-(4-i)}}{q^3+q^{-3}} & \text{if} \quad i > 0 \text{ and } j = -i, \\ \\ (-1)^{i+1}\dfrac{q^{i+4}}{q^3+q^{-3}} & \text{if} \quad i < 0 \text{ and } j = -i. \end{cases}
$$

We are free to scale the bilinear form $(\,|\,)$ on V_ϕ by any nonzero element in $\mathbf{K}$. In writing down the values of $(x_i|x_{-i})$ in Proposition 2.20, we have chosen a particular scaling with an eye towards results (such as Proposition 4.2) to follow. It is easy to see, using the basis elements x_i, that the following holds.

Proposition 2.21. *The transformation $\jmath$ is an isometry ($(x|y) = (\jmath(x)|\jmath(y))$ for all $x, y \in V$) relative to the bilinear form in Proposition 2.20.*

Remark. Observe that $\check{R}_{V,V} \circ (\,|\,) = \xi(\,|\,)$, where ξ is the eigenvalue of $\check{R}_{V,V}$ on $L(0)$. Since $\check{R}_{V,V}(x_1 \otimes x_{-1}) = q(x_{-1} \otimes x_1)$, as our calculations for the "test element" show, we have $\xi(x_1|x_{-1}) = q(x_{-1}|x_1)$. Therefore, $\xi q^{-3} = qq^3$ or $\xi = q^7$, as asserted in (2.15).

For D_4, the half-sum of the positive roots is given by $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6}K_{\alpha_2}^{-10}K_{\alpha_3}^{-6}K_{\alpha_4}^{-6}$. Therefore, on the natural representation V, $K_{2\rho}^{-1}$ has a diagonal matrix relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ with corresponding diagonal entries given by $\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}$. It is easy to verify directly that

$$
(x|y) = (y|K_{2\rho}^{-1}x). \tag{2.22}
$$

3. Octonions and Quantum Octonions

Suppose $\zeta = (1\ 3)$, $\eta = (1\ 4)$, and V_ϕ is the representation of $U_q(\mathrm{D}_4)$ coming from the graph automorphism ϕ. Then

$$
\dim_{\mathbf{K}} \mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta) \otimes \mathcal{S}^2(V_\eta), \mathbf{K}) = 1, \tag{3.1}
$$

and we can realize these homomorphisms explicitly as multiples of the mapping

$$\begin{aligned} &\mathcal{S}^2(V_\zeta)\otimes\mathcal{S}^2(V_\eta)\subset V_\zeta\otimes V_\zeta\otimes V_\eta\otimes V_\eta\to\mathbf{K},\\ &\sum_{k,\ell}u_k\otimes v_k\otimes w_\ell\otimes y_\ell\mapsto\sum_{k,\ell}(u_k|v_k)(w_\ell|y_\ell). \end{aligned}\tag{3.2}$$

On the other hand, the mapping

(3.3)

$$\begin{aligned} &\mathcal{S}^2(V_\zeta)\otimes\mathcal{S}^2(V_\eta)\subset V_\zeta\otimes V_\zeta\otimes V_\eta\otimes V_\eta\to V_\zeta\otimes V_\eta\otimes V_\zeta\otimes V_\eta\to V\otimes V\to\mathbf{K},\\ &\sum_{k,\ell}u_k\otimes v_k\otimes w_\ell\otimes y_\ell\mapsto\sum_{k,\ell}u_k\otimes\check{R}_{V_\zeta,V_\eta}(v_k\otimes w_\ell)\otimes y_\ell\\ &\qquad\mapsto\sum_{k,\ell}\sum_i u_k\cdot w_\ell^{(i)}\otimes v_k^{(i)}\cdot y_\ell\mapsto\sum_{k,\ell}\sum_i(u_k\cdot w_\ell^{(i)}|v_k^{(i)}\cdot y_\ell), \end{aligned}$$

is a $U_q(\mathrm{D}_4)$-module homomorphism, (where "$\cdot$" is the product in Proposition 2.14 and $\check{R}_{V_\zeta,V_\eta}(v_k\otimes w_\ell)=\sum_i w_\ell^{(i)}\otimes v_k^{(i)}$ is the braiding morphism). Thus, it must be a multiple of the mapping in (3.2).

We now consider what would happen if $q=1$, and compute the multiple in that case. Here the braiding morphism may be taken to be $v\otimes w\mapsto w\otimes v$, since that is a module map in the $q=1$ case. We evaluate the mappings at $e\otimes e\otimes e\otimes e$, where $e=x_4-x_{-4}$. Now by Table 2.7, we have

$$\begin{aligned} e^2&=\jmath(e)*\jmath(e)=(x_4-x_{-4})*(x_4-x_{-4})\\ &=q^{\frac12}x_4-q^{-\frac12}x_{-4}, \end{aligned}\tag{3.4}$$

which shows that $e^2=e$ at $q=1$. By Proposition 2.19,

$$(e|e)=-(x_4|x_{-4})-(x_{-4}|x_4)=\frac{2}{q^3+q^{-3}},\tag{3.5}$$

which gives $(e|e)_1=1$ at $q=1$. *(Here and throughout the paper we use $(\,|\,)_1$ to denote the bilinear form at $q=1$.)* Therefore, $(e|e)_1(e|e)_1=1$ and $(e^2|e^2)_1=(e|e)_1=1$. Thus, the two maps in (3.2) and (3.3) are equal at $q=1$. By specializing $\sum_k u_k\otimes v_k=x\otimes x$ and $\sum_\ell w_\ell\otimes y_\ell=y\otimes y$ (which are symmetric elements at $q=1$), we have as a consequence,

$$(x\cdot y|x\cdot y)_1=(x|x)_1(y|y)_1.\tag{3.6}$$

Using the multiplication table (Table 2.7) for $x*y$ and the formula $x\cdot y=\jmath(x)*\jmath(y)$, we can verify that $e=x_4-x_{-4}$ is the unit element relative to the product "$\cdot$" when $q=1$. (For further discussion, see the beginning of Section 5.) Thus, the product $\cdot: V_\zeta\otimes V_\eta\to V$ for $q=1$ gives an 8-dimensional unital composition algebra over $\mathbf{K}$ whose underlying vector space is $V=V_\zeta=V_\eta$. Such an algebra must be the octonions. Thus, we have

Proposition 3.7. *Let V be the 8-dimensional vector space over $\mathbf{K}$ with basis $\{x_{\pm i}\mid i=1,\dots,4\}$. Then V with the product $\cdot$ defined by $x\cdot y=\jmath(x)*\jmath(y)$, where $\jmath$ is as in (2.12) and $*$ is as in Table 2.7, is the algebra of octonions when $q=1$.*

Because of this result we are motivated to make the following definition.

Quantum Octonions.

Definition 3.8. Let V be the 8-dimensional vector space over $\mathbf{K}$ with basis $\{x_{\pm i} \mid i = 1, \ldots, 4\}$. Then V with the product "$\cdot$" defined by

$$x \cdot y = \jmath(x) * \jmath(y),$$

where $\jmath$ is as in (2.12) and $*$ is given by Table 2.7, is the algebra of *quantum octonions*. We denote this algebra by $\mathbb{O}_q = (V, \cdot)$.

The multiplication table for the quantum octonions is obtained from Table 2.7 by interchanging the entries in the columns and rows labelled by x_{-4} and x_4. As a result, we have

TABLE 3.9

$\cdot$	x_1	x_2	x_3	x_{-4}	x_4	x_{-3}	x_{-2}	x_{-1}
x_1	0	0	0	0	$q^{-\frac{3}{2}}x_1$	$q^{-\frac{3}{2}}x_2$	$q^{-\frac{3}{2}}x_3$	$q^{-\frac{3}{2}}x_{-4}$
x_2	0	0	$-q^{-\frac{1}{2}}x_1$	$-q^{-\frac{1}{2}}x_2$	0	0	$q^{-\frac{3}{2}}x_4$	$q^{-\frac{3}{2}}x_{-3}$
x_3	0	$q^{\frac{1}{2}}x_1$	0	$-q^{-\frac{1}{2}}x_3$	0	$-q^{-\frac{1}{2}}x_4$	0	$q^{-\frac{3}{2}}x_{-2}$
x_4	0	$q^{\frac{1}{2}}x_2$	$q^{\frac{1}{2}}x_3$	0	$q^{\frac{1}{2}}x_4$	0	0	$q^{-\frac{3}{2}}x_{-1}$
x_{-4}	$-q^{\frac{3}{2}}x_1$	0	0	$-q^{-\frac{1}{2}}x_{-4}$	0	$-q^{-\frac{1}{2}}x_{-3}$	$-q^{-\frac{1}{2}}x_{-2}$	0
x_{-3}	$-q^{\frac{3}{2}}x_2$	0	$q^{\frac{1}{2}}x_{-4}$	0	$q^{\frac{1}{2}}x_{-3}$	0	$-q^{-\frac{1}{2}}x_{-1}$	0
x_{-2}	$-q^{\frac{3}{2}}x_3$	$-q^{\frac{3}{2}}x_{-4}$	0	0	$q^{\frac{1}{2}}x_{-2}$	$q^{\frac{1}{2}}x_{-1}$	0	0
x_{-1}	$-q^{\frac{3}{2}}x_4$	$-q^{\frac{3}{2}}x_{-3}$	$-q^{\frac{3}{2}}x_{-2}$	$-q^{\frac{3}{2}}x_{-1}$	0	0	0	0

The product map: $p : V_\zeta \otimes V_\eta \to V$, $p(x \otimes y) = x \cdot y$, of the quantum octonion algebra is the composition $p = * \circ (j \otimes j)$ of two $U_q(\mathrm{D}_4)$-module maps, so is itself a $U_q(\mathrm{D}_4)$-module homomorphism.

Let $\pi : U_q(\mathrm{D}_4) \to \mathrm{End}(V)$ be the natural representation of $U_q(\mathrm{D}_4)$. Then we have

$$\begin{aligned} \pi(a)(x \cdot y) &= a \cdot p(x \otimes y) = p(\Delta(a)(x \otimes y)) \\ &= p\Big(\sum_a \zeta(a_{(1)})x \otimes \eta(a_{(2)})y\Big) \\ &= \sum_a \jmath(\zeta(a_{(1)}x)) * \jmath(\eta(a_{(2)}y)) \\ &= \sum_a \Big(\zeta(a_{(1)})x\Big) \cdot \Big(\eta(a_{(2)})y\Big). \end{aligned} \tag{3.10}$$

Now when $q = 1$ we can identify $U_q(\mathrm{D}_4)$ with the universal enveloping algebra $U(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ of type D_4. The comultiplication specializes in this case to the usual comultiplication on $U(\mathfrak{g})$, which has $\Delta(a) = a \otimes 1 + 1 \otimes a$ for all $a \in \mathfrak{g}$. In particular, for elements $a \in \mathfrak{g}$, (3.10) becomes

$$\pi(a)(x \cdot y) = \big(\zeta(a)x\big) \cdot y + x \cdot \big(\eta(a)y\big). \tag{3.11}$$

Equation (3.11) is commonly referred to as the *Principle of Local Triality* (see for example, [J, p. 8] or [S, p. 88]). As a result, we have

Proposition 3.12. *The quantum octonion algebra* $\mathbb{O}_q = (V, \cdot)$ *satisfies the q-Principle of Local Triality,*

$$\pi(a)(x \cdot y) = \sum_a \Big(\zeta(a_{(1)})x\Big) \cdot \Big(\eta(a_{(2)})y\Big)$$

for all $a \in U_q(\mathrm{D}_4)$ *and* $x, y \in V$, *where* ζ *and* η *are the graph automorphisms* $\zeta = (1\ 3)$ *and* $\eta = (1\ 4)$ *of* $U_q(\mathrm{D}_4)$, *and* $\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}$ *is the comultiplication in* $U_q(\mathrm{D}_4)$.

A quantum para-Hurwitz algebra. At $q = 1$, the map $\jmath : V \to V$ sending $x_i \mapsto -x_{-i}$ for $i \neq \pm 4$, $x_4 \mapsto -x_{-4}$, and $x_{-4} \mapsto -x_4$ agrees with the standard involution $x \to \overline{x}$ on the octonions, so that

$$x * y = \jmath(x) \cdot \jmath(y) = \overline{x} \cdot \overline{y}. \tag{3.13}$$

Thus, the product "$*$" displayed in Table 2.7 gives what is called the *para-Hurwitz algebra* when $q = 1$. (Results on para-Hurwitz algebras can be found in [EM], [EP], [OO1], [OO2].)

Definition 3.14. We say that the algebra $\mathbb{P}_q = (V, *)$ with the multiplication $*$ given by Table 2.7 is the *quantum para-Hurwitz algebra*. It satisfies

$$x * y = \jmath(x) \cdot \jmath(y),$$

where $\mathbb{O}_q = (V, \cdot)$ is the quantum octonion algebra and $\jmath$ is the transformation on V defined in (2.12).

Our definitions have been predicated on using the mapping $\cdot : V_\zeta \otimes V_\eta \to V$. We can ask what would have happened if we had used the projection $V_\eta \otimes V_\zeta \to V$ instead. To answer this question, it is helpful to observe that the mapping $\jmath$ determines a $U_q(\mathrm{D}_4)$-module isomorphism $\jmath : V_{\zeta\eta\zeta} = V_{(3\ 4)} \to V$, which is readily apparent from comparing the following picture of $V_{(3\ 4)}$ with the diagram for V in (2.6):

$V_{(3\ 4)}$

$x_1 \circ$
$\downarrow 1$
$x_2 \circ$
$\downarrow 2$
$x_3 \circ$
$4 \swarrow \qquad \searrow 3$
$x_4 \circ \qquad \circ\, x_{-4}$
$3 \searrow \qquad \swarrow 4$
$x_{-3} \circ$
$\downarrow 2$
$x_{-2} \circ$
$\downarrow 1$
$x_{-1} \circ$

Suppose as before that $p : V_\zeta \otimes V_\eta \to V$ is the product $p(x \otimes y) = x \cdot y$ in the quantum octonion algebra. Consider the mapping $\jmath \circ p \circ (\jmath \otimes \jmath) : V_\eta \otimes V_\zeta \to V$. We demonstrate that this is a $U_q(\mathrm{D}_4)$-module map. Bear in mind in these calculations that $\jmath : V_\zeta \to V_- = V_{\zeta\eta}$ and $\jmath : V_\eta \to V_+ = V_{\eta\zeta}$ for $\zeta = (1\ 3)$ and $\eta = (1\ 4)$. Then for $a \in U_q(\mathrm{D}_4)$ we have

$$\begin{aligned}\jmath \circ p \circ (\jmath \otimes \jmath)(a(x \otimes y)) &= \jmath \circ p \circ (\jmath \otimes \jmath)\Big(\sum_a \eta(a_{(1)})x \otimes \zeta(a_{(2)})y\Big)\\ &= \jmath \circ p\Big(\sum_a j(\eta(a_{(1)})x) \otimes j(\zeta(a_{(2)})y)\Big)\\ &= \jmath \circ p\Big(\sum_a \eta\zeta(a_{(1)})j(x) \otimes \zeta\eta(a_{(2)})j(y)\Big)\\ &= \jmath \circ p\Big(\sum_a \zeta^2\eta\zeta(a_{(1)})j(x) \otimes \eta^2\zeta\eta(a_{(2)})j(y)\Big)\end{aligned}$$

Now, using the fact that $\eta\zeta\eta = \zeta\eta\zeta$ and that $(\phi \otimes \phi)\Delta = \Delta\phi$ for all graph automorphisms ϕ together with (3.10), we see that the last expression can be rewritten as

$$\begin{aligned}&\jmath \circ p\Big((\zeta \otimes \eta)\big(\Delta(\zeta\eta\zeta(a))\big)\big(\jmath(x) \otimes \jmath(y)\big)\Big)\\ &\qquad = \jmath\Big(\zeta\eta\zeta(a)p\big(\jmath(x) \otimes \jmath(y)\big)\Big) \quad (p \text{ is a } U_q(\mathrm{D}_4)\text{-homomorphism})\\ &\qquad = a(\jmath \circ p \circ (\jmath \otimes \jmath))(x \otimes y).\end{aligned}$$

Thus, $\jmath \circ p \circ (\jmath \otimes \jmath) : V_\eta \otimes V_\zeta \to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism. Since the space $\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\eta \otimes V_\zeta, V)$ is one-dimensional, we may assume the product $p' : V_\eta \otimes V_\zeta \to V$ is given by the map we have just found, so that $p' = \jmath \circ p \circ (\jmath \otimes \jmath)$. Alternately, $\jmath \circ p' = p \circ (\jmath \otimes \jmath)$. This says that had we defined the quantum octonions using the product p' rather than p so that $x \cdot' y = p'(x \otimes y)$, the two algebras would be isomorphic via the map $\jmath$.

4. Properties of Quantum Octonions

We use $U_q(\mathrm{D}_4)$-module homomorphisms to deduce various properties of the quantum octonions. At $q = 1$ these properties specialize to well-known identities satisfied by the octonions, which can be found for example in [ZSSS]. Since the vector spaces V, V_ζ, and V_η are the same, we will denote the identity map on them simply by "id" in what follows.

Proposition 4.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{S}^2(V_\zeta) \otimes V_\eta & \overset{(\,|\,) \otimes \mathrm{id}}{\longrightarrow} & V_\eta \\ \mathrm{id} \otimes p \downarrow & & \downarrow \mathrm{id} \\ V_\zeta \otimes V & \overset{p \circ (\jmath \otimes \mathrm{id})}{\longrightarrow} & V_\eta, \end{array}$$

so that

$$\sum_k \jmath(u_k) \cdot \big(v_k \cdot y\big) = \sum_k (u_k|v_k)y \tag{4.2}$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q)$, $y \in \mathbb{O}_q$. *In particular, at $q = 1$ this reduces to the well-known identity satisfied by the octonion algebra $\mathbb{O}$:*

$$\overline{x} \cdot \big(x \cdot y\big) = (x|x)_1 y. \tag{4.3}$$

Proof. Since $p : V_\zeta \otimes V_\eta \to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism, we have that $\pi(a) \circ p = p\Big((\zeta \otimes \eta)\Delta(a)\Big)$, which implies that $\pi(\eta(a)) \circ p = p\Big((\zeta \otimes \eta)\Delta(\eta a)\Big) = p\Big((\zeta\eta \otimes \mathrm{id})\Delta(a)\Big)$. This says that $p : V_{\zeta\eta} \otimes V \to V_\eta$ is also a $U_q(\mathrm{D}_4)$-module homomorphism. Since $\jmath : V_\zeta \to V_{\zeta\eta}$ is a $U_q(\mathrm{D}_4)$-module homomorphism, it follows that $\big(p\circ(\jmath\otimes\mathrm{id})\big)\circ(\mathrm{id}\otimes p) \in \mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta)\otimes V_\eta, \mathbf{K})$. As that space of homomorphisms has dimension one, the maps in the diagram must be proportional.

To find the proportionality constant, let us consider the actions using the test element:

$$(x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1 \overset{(\,|\,)\otimes\mathrm{id}}{\to} x_1$$

and

$$\begin{aligned} p \circ (\jmath \otimes \mathrm{id}) \circ (\mathrm{id} \otimes p)\Big((x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1\Big) \\ = p \circ (\jmath \otimes \mathrm{id})\Big(- q^{\frac{3}{2}} x_1 \otimes x_4\Big) = q^{\frac{3}{2}} x_1 \cdot x_4 = x_1. \end{aligned}$$

Therefore, the two maps are equal as claimed. At $q = 1$ we may substitute $x \otimes x \in \mathcal{S}^2(\mathbb{O})$ and $y \in \mathbb{O}$ into the above identity to obtain the corresponding one for the octonions. □

Proposition 4.4. *The following diagram commutes:*

$$\begin{array}{ccc} V_\zeta \otimes \mathcal{S}^2(V_\eta) & \overset{\mathrm{id}\otimes(\,|\,)}{\longrightarrow} & V_\zeta \\ p\otimes\mathrm{id}\downarrow & & \downarrow\mathrm{id} \\ V \otimes V_\eta & \overset{p\circ(\mathrm{id}\otimes\jmath)}{\longrightarrow} & V_\zeta, \end{array}$$

so that

$$\sum_k \big(y \cdot u_k\big) \cdot \jmath(v_k) = \sum_k (u_k|v_k) y \tag{4.5}$$

for all $\sum_k u_k \otimes v_k \in \mathcal{S}^2(\mathbb{O}_q)$, $y \in \mathbb{O}_q$. *In particular, at* $q = 1$ *this gives the identity*

$$\big(y \cdot x\big) \cdot \overline{x} = (x|x)_1 y, \tag{4.6}$$

which is satisfied by the octonions.

Proof. Both maps are $U_q(\mathrm{D}_4)$-module homomorphisms and so must be multiples of each other. We compute their images on $x = x_1 \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1)$:

$$\begin{aligned} &\big(\mathrm{id} \otimes (\,|\,)\big)x = \Big((x_1|x_{-1}) + (x_{-1}|x_1)\Big)x_1 = x_1, \\ &p \circ (\mathrm{id} \otimes \jmath) \circ (p \otimes \mathrm{id})x = p \circ (\mathrm{id} \otimes \jmath)(q^{-\frac{3}{2}} x_{-4} \otimes x_1) = -q^{-\frac{3}{2}} x_{-4} \cdot x_1 = x_1. \end{aligned}$$

Hence we have the desired equality. □

Proposition 4.7. *The following diagram commutes:*

$$\begin{array}{ccc} V_\zeta \otimes V_\eta \otimes V & \overset{p\otimes \mathrm{id}}{\longrightarrow} & V \otimes V \\ \mathrm{id}\otimes(\jmath\circ p\circ(\mathrm{id}\otimes\jmath))\downarrow & & \downarrow (\,|\,) \\ V_\zeta\otimes V_\zeta & \overset{(\,|\,)}{\longrightarrow} & \mathbf{K} \end{array}$$

so that

$$(x\cdot y|z) = (x|\jmath(y\cdot\jmath(z))) = (\jmath(x)|y\cdot\jmath(z)) \tag{4.8}$$

for all x, y, z in the quantum octonions. At $q = 1$ this reduces to the identity

$$(x\cdot y|z)_1 = (x|z\cdot\overline{y})_1, \tag{4.9}$$

satisfied by the octonions.

Proof. Recall we have shown that $\jmath\circ p\circ(\jmath\otimes\jmath) : V_\eta\otimes V_\zeta\to V$ is a $U_q(\mathrm{D}_4)$-module homomorphism, which is equivalent to saying that

$$a\circ\jmath\circ p\circ(\jmath\otimes\jmath) = \jmath\circ p\circ(\jmath\otimes\jmath)\Big((\eta\otimes\zeta)\Delta(a)\Big)$$

for all $a\in U_q(\mathrm{D}_4)$. This implies

$$\begin{aligned}\zeta(a)\circ\jmath\circ p\circ(\jmath\otimes\jmath) &= \jmath\circ p\circ(\jmath\otimes\jmath)\Big((\eta\otimes\zeta)\Delta(\zeta(a))\Big)\\ &= \jmath\circ p\circ(\jmath\otimes\jmath)\Big((\eta\otimes\zeta)(\zeta\otimes\zeta)\Delta(a)\Big)\\ &= \jmath\circ p\circ(\jmath\otimes\jmath)\Big((\eta\zeta\otimes\mathrm{id})\Delta(a)\Big),\end{aligned}$$

so that $\jmath\circ p\circ(\jmath\otimes\jmath) : V_{\eta\zeta}\otimes V\to V_\zeta$ is a $U_q(\mathrm{D}_4)$-module homomorphism. From this we deduce that

$$\jmath\circ p\circ(\mathrm{id}\otimes\jmath) = \jmath\circ p\circ(\jmath\otimes\jmath)\circ(\jmath\otimes\mathrm{id}) : V_\eta\otimes V\to V_\zeta$$

is a $U_q(\mathrm{D}_4)$-module homomorphism as well. Thus, the mappings in the above diagram are scalar multiples of one another. The scalar can be discovered by computing the values of them on $x_1\otimes x_{-3}\otimes x_{-2}$:

$$(x_1\cdot x_{-3}|x_{-2}) = q^{-\frac{3}{2}}(x_2|x_{-2}) = \frac{-q^{-\frac{3}{2}}q^{-2}}{q^3+q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3+q^{-3}},$$

$$(x_1|\jmath(x_{-3}\cdot\jmath(x_{-2}))) = (x_1|\jmath(q^{-\frac{1}{2}}x_{-1})) = \frac{-q^{-\frac{1}{2}}q^{-3}}{q^3+q^{-3}} = \frac{-q^{-\frac{7}{2}}}{q^3+q^{-3}}.$$

Hence, they agree, and we have the result. The fact that $\jmath$ is an isometry gives the last equality in (4.8). □

Proposition 4.10. $(x\cdot y|z) = (\jmath(y)|\jmath(z)\cdot K_{2\rho}^{-1}x)$ *for all $x, y, z\in\mathbb{O}_q = (V,\cdot)$, which at $q = 1$ gives*

$$(x\cdot y|z)_1 = (y|\overline{x}\cdot z)_1 \tag{4.11}$$

for the octonions.

Proof. This is an immediate consequence of the calculation

$$(x \cdot y|z) \overset{1}{=} (x|\jmath(y \cdot \jmath(z))) \overset{2}{=} (\jmath(y \cdot \jmath(z))|K_{2\rho}^{-1}x)$$
$$= (y \cdot \jmath(z)|\jmath(K_{2\rho}^{-1}x)) \overset{3}{=} (\jmath(y)|\jmath(z) \cdot K_{2\rho}^{-1}x),$$

where (1) is from Proposition 4.7, (2) uses (2.22), and (3) is Proposition 4.7. □

Note that $K_{2\rho}^{-1}$ acts on $\mathbb{O}_q$ as an algebra automorphism because it is group-like.

Proposition 4.12. *The following diagram commutes:*

$$\begin{array}{ccc} V_\eta \otimes \mathcal{S}^2(V_\zeta) \otimes V_\eta & \xrightarrow{\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}} & V_\eta \otimes V_\eta \\ (\jmath \circ p \circ (\jmath \otimes \jmath)) \otimes p \downarrow & & \downarrow (\,|\,) \\ V \otimes V & \xrightarrow{(\,|\,)} & \mathbf{K}. \end{array}$$

When $q = 1$, this gives the identity

$$(x \cdot y|x \cdot z)_1 = (x|x)_1 (y|z) \tag{4.13}$$

for the octonions.

Proof. First, it is helpful to observe that

$$\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\eta \otimes \mathcal{S}^2(V_\zeta) \otimes V_\eta, \mathbf{K}) \cong \mathrm{Hom}_{U_q(\mathrm{D}_4)}(\mathcal{S}^2(V_\zeta), V_\eta \otimes V_\eta)$$
$$= \mathrm{Hom}_{U_q(\mathrm{D}_4)}(L(0), L(0)),$$

so these spaces have dimension 1. We evaluate the mappings on $x = x_{-1} \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$:

$$(\,|\,) \circ \Big(\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}\Big)(x) = (x_{-1}|x_1) = \frac{q^3}{q^3 + q^{-3}},$$
$$(\,|\,) \circ \Big(\big(\jmath \circ p \circ (\jmath \otimes \jmath)\big) \otimes p\Big)(x) = (\jmath(x_{-1} \cdot x_1)|x_{-1} \cdot x_1)$$
$$= -q^3(x_{-4}|x_4) = \frac{q^3}{q^3 + q^{-3}}. \qquad \square$$

Proposition 4.14. *The following diagram commutes:*

$$\begin{array}{ccc} V_\zeta \otimes \mathcal{S}^2(V_\eta) \otimes V_\zeta & \xrightarrow{\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}} & V_\zeta \otimes V_\zeta \\ p \otimes (\jmath \circ p \circ (\jmath \otimes \jmath)) \downarrow & & \downarrow (\,|\,) \\ V \otimes V & \xrightarrow{(\,|\,)} & \mathbf{K} \end{array}$$

In particular, at $q = 1$ this gives the following identity satisfied by the octonions:

$$(y \cdot x|z \cdot x)_1 = (y|z)_1 (x|x)_1. \tag{4.15}$$

Proof. The proof is virtually identical to that of Proposition 4.12. We check the maps on $x = x_{-1} \otimes (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$:

$$(\,|\,) \circ \Big(p \otimes \big(\jmath \circ p \circ (\jmath \otimes \jmath)\big)\Big)(x) = (x_{-1} \cdot x_1|\jmath(x_{-1} \cdot x_1)) = (\jmath(x_{-1} \cdot x_1)|x_{-1} \cdot x_1)$$
$$= -q^3(x_{-4}|x_4) = \frac{q^3}{q^3 + q^{-3}},$$

which equals $(\,|\,) \circ \big(\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}\big)(x)$. □

r-algebras. The notion of an r-algebra (that is, an algebra equipped with a Yang-Baxter operator) arises in Manin's work [M] on noncommutative geometry. Noteworthy examples of r-algebras are Weyl and Clifford algebras, noncommutative tori, certain universal enveloping algebras, and quantum groups (see, for example, Baez [B]).

Definition 4.16. Suppose A is an algebra over a field $\mathbf{F}$ with multiplication $p : A \otimes A \to A$. Assume $R \in \mathrm{End}(A^{\otimes 2})$ is a Yang-Baxter operator, i.e. $R_{1,2}R_{2,3}R_{1,2} = R_{2,3}R_{1,2}R_{2,3}$, where $R_{1,2} = R \otimes \mathrm{id} \in \mathrm{End}(A^{\otimes 3})$ and $R_{2,3} = \mathrm{id} \otimes R \in \mathrm{End}(A^{\otimes 3})$. Then A is said to be an r-algebra if

$$\begin{aligned} R \circ (p \otimes \mathrm{id}_A) &= (\mathrm{id}_A \otimes p) \circ R_{1,2} \circ R_{2,3}, \\ R \circ (\mathrm{id}_A \otimes p) &= (p \otimes \mathrm{id}_A) \circ R_{2,3} \circ R_{1,2}. \end{aligned} \tag{4.17}$$

When A has a unit element 1, the Yang-Baxter operator R is also required to satisfy $R(1 \otimes x) = x \otimes 1$ and $R(x \otimes 1) = 1 \otimes x$ for all $x \in A$.

Our quantum octonions satisfy r-algebra properties similar to those in (4.17) with respect to the R-matrix of $U_q(\mathrm{D}_4)$.

Proposition 4.18. *(a) In* $\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, V \otimes V)$ *we have*

$$\check{R}_{V,V} \circ (p \otimes \mathrm{id}) = (\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}.$$

(b) In $\mathrm{Hom}_{U_q(\mathrm{D}_4)}(V \otimes V_\zeta \otimes V_\eta, V \otimes V)$ *we have*

$$\check{R}_{V,V} \circ (\mathrm{id} \otimes p) = (p \otimes \mathrm{id}) \circ \check{R}_{V,V_\eta} \circ \check{R}_{V,V_\zeta}.$$

Proof. Suppose that $\pi_1 : V \otimes V \to L(2\omega_1)$, $\pi_2 : V \otimes V \to L(\omega_2)$, and $\pi_3 : V \otimes V \to L(0)$ are the projections of $V \otimes V$ onto its irreducible summands. Let T_ℓ and T_r denote the transformations on the left and right in (a). We compute T_ℓ and T_r on several elements of $V_\zeta \otimes V_\eta \otimes V$ which have been chosen so that Θ acts as the identity. We use the fact that $\eta(\epsilon_1) = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)$, $\zeta(\epsilon_1) = \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)$, $\eta(\epsilon_2) = \frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)$, and $\zeta(\epsilon_3) = \frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4)$.

We begin by evaluating the action of T_ℓ and T_r on $x_{-1} \otimes x_1 \otimes x_{-1}$:

$$\Big(\check{R}_{V,V} \circ (p \otimes \mathrm{id})\Big)(x_{-1} \otimes x_1 \otimes x_{-1}) = -q^{\frac{3}{2}}\check{R}_{V,V}(x_4 \otimes x_{-1}) = -q^{\frac{3}{2}}(x_{-1} \otimes x_4),$$

$$\begin{aligned} &\Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V} \circ \check{R}_{V_\eta,V}\Big)(x_{-1} \otimes x_1 \otimes x_{-1}) \\ &\quad = q^{\frac{1}{2}}\Big((\mathrm{id} \otimes p) \circ \check{R}_{V_\zeta,V}\Big)(x_{-1} \otimes x_{-1} \otimes x_1) \\ &\quad = (\mathrm{id} \otimes p)(x_{-1} \otimes x_{-1} \otimes x_1) \\ &\quad = -q^{\frac{3}{2}}(x_{-1} \otimes x_4). \end{aligned}$$

Now $(x_{-1}|x_4) = 0$ implies that $x_{-1} \otimes x_4$ lies in $L(2\omega_1) \oplus L(\omega_2)$. The vector $x_{-1} \otimes x_4$ is not an eigenvector of $\check{R}_{V,V}$, because $\check{R}_{V,V}(x_4 \otimes x_{-1}) = x_{-1} \otimes x_4$. Therefore, $\pi_1(x_{-1} \otimes x_4) \neq 0$ and $\pi_2(x_{-1} \otimes x_4) \neq 0$. Since $\dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(2\omega_1)) = 1 = \dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta \otimes V_\eta \otimes V, L(\omega_2))$, we see from the above calculation that $\pi_1 T_\ell = \pi_1 T_r$ and $\pi_2 T_\ell = \pi_2 T_r$.

Consider the action of T_ℓ and T_r on $x_3 \otimes x_2 \otimes x_{-1}$:

$$\Big(\check{R}_{V,V} \circ (p \otimes \mathrm{id})\Big)(x_3 \otimes x_2 \otimes x_{-1}) = q^{\frac{1}{2}}\check{R}_{V,V}(x_1 \otimes x_{-1}) = q^{\frac{3}{2}}(x_{-1} \otimes x_1),$$

$$
\begin{aligned}
\Big((\mathrm{id}\otimes p)\circ \check{R}_{V_\zeta,V}\circ \check{R}_{V_\eta,V}\Big)&(x_3\otimes x_2\otimes x_{-1})\\
&= q^{\frac12}\Big((\mathrm{id}\otimes p)\circ \check{R}_{V_\zeta,V}\Big)(x_3\otimes x_{-1}\otimes x_2)\\
&= q(\mathrm{id}\otimes p)(x_{-1}\otimes x_3\otimes x_2)\\
&= q^{\frac32}(x_{-1}\otimes x_1).
\end{aligned}
$$

The projection π_3 comes from the form $(\,|\,)$, and so it is nonzero when it acts on $x_{-1}\otimes x_1$. Consequently, applying π_3 to both sides of $T_\ell(x_3\otimes x_2\otimes x_{-1}) = T_r(x_3\otimes x_2\otimes x_{-1})$ and using $\dim \mathrm{Hom}_{U_q(\mathrm{D}_4)}(V_\zeta\otimes V_\eta\otimes V, L(0)) = 1$, we obtain $\pi_3 T_\ell = \pi_3 T_r$. Therefore, $T_\ell = (\pi_1+\pi_2+\pi_3)T_\ell = (\pi_1+\pi_2+\pi_3)T_r = T_r$. The calculations for part (b) are almost identical and are left as an exercise for the reader. □

5. Properties of the Quantum Para-Hurwitz Algebra

Recall that the quantum para-Hurwitz algebra $\mathbb{P}_q = (V,*)$ is obtained from identifying the underlying vector spaces of V_θ, V_{θ^2} and V, where V is the natural representation of $U_q(\mathrm{D}_4)$ and θ is the graph automorphism $\theta = (1\,4\,3)$, and using the multiplication $p^*: V_\theta\otimes V_{\theta^2}\to V$, $x\otimes y\mapsto x*y = \jmath(x)\cdot\jmath(y)$. The resulting product is displayed in Table 2.7. As we saw in the last section, the quantum octonion algebra enjoys very nice composition properties relative to the bilinear form $(\,|\,)$. It is known that the para-Hurwitz algebra $\mathbb{P}$ (that is the $q=1$ version of $\mathbb{P}_q$), which is obtained from the octonions using the product $x*y = \overline{x}\cdot\overline{y}$, has an associative bilinear form admitting composition. Here we establish similar results for the bilinear form on $\mathbb{P}_q = (V,*)$

Proposition 5.1. *For the quantum para-Hurwitz algebra $\mathbb{P}_q = (V,*)$,*

$$(x*y|z) = (x|y*z) \qquad \text{and} \qquad (x*y|z) = (y|z*K_{2\rho}^{-1}x)$$

for all $x,y,z\in\mathbb{P}_q$. At $q=1$ these relations reduce to well-known properties of the para-Hurwitz algebra: $(x*y|z)_1 = (x|y*z)_1$ and $(x*y|z)_1 = (y|z*x)_1$.

Proof. The proof amounts to using the fact that $\jmath$ is an isometry together with the fact that $\jmath$ commutes with $K_{2\rho}^{-1}$. So we have

$$
\begin{aligned}
(x*y|z) &= (\jmath(x)\cdot\jmath(y)|z)\\
&= \begin{cases} (x|\jmath(y)\cdot\jmath(z)) = (x|y*z),\\ (y|\jmath(z)\cdot K_{2\rho}^{-1}\jmath(x)) = (y|\jmath(z)\cdot\jmath(K_{2\rho}^{-1}x)) = (y|z*K_{2\rho}^{-1}x). \end{cases}
\end{aligned}
$$
□

Proposition 5.2. *Ignoring the $U_q(\mathrm{D}_4)$-module structures, we have*

$$
\begin{array}{ccc}
V\otimes\mathcal{S}^2(V)\otimes V & \overset{\mathrm{id}\otimes(\,|\,)\otimes\mathrm{id}}{\longrightarrow} & V\otimes V\\
(\jmath\circ p^*\circ(\jmath\otimes\jmath))\otimes p^*\downarrow & & \downarrow(\,|\,)\\
V\otimes V & \overset{(\,|\,)}{\longrightarrow} & \mathbf{K}
\end{array}
$$

and

$$\begin{array}{ccc} V \otimes \mathcal{S}^2(V) \otimes V & \xrightarrow{\mathrm{id} \otimes (\,|\,) \otimes \mathrm{id}} & V \otimes V \\ p^* \otimes (\jmath \circ p^* \circ (\jmath \otimes \jmath)) \downarrow & & \downarrow (\,|\,) \\ V \otimes V & \xrightarrow{(\,|\,)} & \mathbf{K} \end{array}$$

Proof. These diagrams commute when p is used in place of p^*. Now when $p^* = p \circ (\jmath \otimes \jmath)$ is taken, we have

$$\begin{aligned} (\,|\,) \circ (\jmath \circ p^* \circ (\jmath \otimes \jmath)) \otimes p^* &= (\,|\,) \circ \Big((\jmath \circ p) \otimes (p \circ (\jmath \otimes \jmath)) \Big) \\ &\overset{1}{=} (\,|\,) \circ p \otimes (j \circ p \circ (\jmath \otimes \jmath)) \\ &= (\,|\,) \circ \mathrm{id} \otimes (\,|\,) \otimes \mathrm{id} \end{aligned}$$

where (1) follows from the fact that $\jmath$ is an isometry. The other diagram can be shown to commute in a similar way. □

Proposition 5.3. *The following diagram commutes and shows at $q = 1$ that the relation $(x \cdot y) \cdot x = (x|x)_1 y = x \cdot (y \cdot x)$ holds for the para-Hurwitz algebra:*

$$\begin{array}{ccccc} V_{\theta^2} \otimes V & \xleftarrow{\mathrm{id} \otimes (\jmath \circ p^* \circ (\jmath \otimes \jmath))} & \mathcal{S}^2(V_{\theta^2}) \otimes V_\theta = \mathcal{S}^2(V) \otimes V = \mathcal{S}^2(V_\theta) \otimes V_{\theta^2} & \xrightarrow{\mathrm{id} \otimes p^*} & V_\theta \otimes V \\ \downarrow_{p^*} & & {\scriptstyle (\,|\,) \otimes \mathrm{id}} \downarrow & & {\scriptstyle \jmath \circ p^* \circ (\jmath \otimes \jmath)} \downarrow \\ V_\theta & = & V & = & V_{\theta^2} \end{array}$$

Proof. Again it suffices to check the maps on $x = (x_1 \otimes x_{-1} + x_{-1} \otimes x_1) \otimes x_1$, and for it we have

$$\begin{aligned} \Big(\jmath \circ p^* \circ (\jmath \otimes \jmath)\Big) \circ (\mathrm{id} \otimes p^*)(x) &= \jmath \circ p^* \circ (\jmath \otimes \jmath)(x_1 \otimes (-q^{\frac{3}{2}}) x_4) \\ &= -\jmath(x_1) = x_1 = (\,|\,) \otimes \mathrm{id}(x), \end{aligned}$$

while

$$p^* \circ \Big(\mathrm{id} \otimes \big(\jmath \circ p^* \circ (\jmath \otimes \jmath) \big) \Big)(x) = p^*(x_1 \otimes q^{\frac{3}{2}} x_{-4}) = x_1 = (\,|\,) \otimes \mathrm{id}(x). \qquad \square$$

6. Idempotents and Derivations of Quantum Octonions

It is apparent from Table 3.9 that $e_1 = q^{-\frac{1}{2}} x_4$ and $e_2 = -q^{\frac{1}{2}} x_{-4}$ are orthogonal idempotents in the quantum octonion algebra $\mathbb{O}_q = (V, \cdot)$. Their sum $e = e_1 + e_2$ acts as a left and right identity element on the span of $e_1, e_2, x_j, x_{-j}, j = 2, 3$. Moreover, the relations $e \cdot x_1 = q^2 x_1$, $x_1 \cdot e = q^{-2} x_1$, $e \cdot x_{-1} = q^{-2} x_{-1}$, and $x_{-1} \cdot e = q^2 x_{-1}$ hold in $\mathbb{O}_q$. Hence, at $q = 1$ or -1, the element e is an identity element. Relative to the basis $e_1, e_2, x_{\pm j}, j = 1, 2, 3$, the left and right multiplication operators $L_{e_1}, R_{e_1}, L_{e_2}, R_{e_2}$ can be simultaneously diagonalized. The operators L_{e_1}, R_{e_1} determine a Peirce decomposition of $\mathbb{O}_q$,

(6.1)
$$\mathbb{O}_q = \oplus_{(\gamma,\delta)} (\mathbb{O}_q)_{\gamma,\delta} \quad \text{where} \quad (\mathbb{O}_q)_{\gamma,\delta} = \{ v \in \mathbb{O}_q \mid e_1 \cdot v = \gamma v, \text{ and } v \cdot e_1 = \delta v \}.$$

Now $(\mathbb{O}_q)_{0,q^{-2}} = \mathbf{K}x_1$, $(\mathbb{O}_q)_{1,0} = \mathbf{K}x_2 + \mathbf{K}x_3$, $(\mathbb{O}_q)_{1,1} = \mathbf{K}e_1$, $(\mathbb{O}_q)_{0,0} = \mathbf{K}e_2$, $(\mathbb{O}_q)_{0,1} = \mathbf{K}x_{-2} + \mathbf{K}x_{-3}$, and $(\mathbb{O}_q)_{q^{-2},0} = \mathbf{K}x_{-1}$. The idempotent e_2 gives a similar Peirce decomposition.

Suppose d is a derivation of $\mathbb{O}_q$ such that $d(e_1) = 0 = d(e_2)$. It is easy to see that d must map each Peirce space $(\mathbb{O}_q)_{\gamma,\delta}$ into itself. Thus, we may suppose that

$$\begin{aligned} d(x_1) &= b_1x_1, & d(x_{-1}) &= b_{-1}x_{-1}, \\ d(x_2) &= c_{2,2}x_2 + c_{3,2}x_3, & d(x_3) &= c_{2,3}x_2 + c_{3,3}x_3, \\ d(x_{-2}) &= d_{2,2}x_{-2} + d_{3,2}x_{-3}, & d(x_{-3}) &= d_{2,3}x_{-2} + d_{3,3}x_{-3}. \end{aligned} \tag{6.2}$$

Applying d to the relation $x_1 \cdot x_{-1} = q^{-\frac{3}{2}}x_{-4} = -q^{-2}e_2$ (or to the relation $x_{-1} \cdot x_1 = -q^{\frac{3}{2}}x_4 = -q^2e_1$) shows that $b_{-1} = -b_1$. Now from $x_k \cdot x_k = 0$, we deduce that $c_{j,k} = 0$ for $j \neq k$, and similarly from $x_{-k} \cdot x_{-k} = 0$ it follows that $d_{j,k} = 0$. There are relations $x_1 \cdot x_{-j} = \xi x_k$, $x_{-j} \cdot x_1 = \xi x_k$, $x_{-1} \cdot x_j = \xi x_{-k}$, and $x_j \cdot x_{-1} = \xi x_{-k}$ (where $j \neq k$ and ξ in each one indicates some appropriate scalar that can be found in Table 3.9), and applying d to those equations gives $b_1 + d_{j,j} = c_{k,k}$ and $b_{-1} + c_{j,j} = d_{k,k}$. The second is equivalent to the first since $b_{-1} = -b_1$. From $x_{\pm j} \cdot x_{\pm k} = \xi x_{\pm 1}$ we see that $c_{2,2} + c_{3,3} = b_1$ and $d_{2,2} + d_{3,3} = -b_1$. Finally, applying d to $x_j \cdot x_{-j} = \xi e_1$ or $x_{-j} \cdot x_j = \xi e_2$, we obtain $d_{j,j} = -c_{j,j}$ for $j = 2, 3$. Every nonzero product between basis elements in the quantum octonions is one of these types, or it is one involving the idempotents e_1, e_2, so we have determined all the possible relations. Consequently, we can conclude that for any derivation d such that $d(e_1) = 0 = d(e_2)$, there are scalars $b = b_1$ and $c = c_{2,2}$ so that the following hold:

$$\begin{aligned} & d(e_1) = 0 = d(e_2), \\ d(x_1) &= bx_1, & d(x_{-1}) &= -bx_{-1}, \\ d(x_2) &= cx_2, & d(x_{-2}) &= -cx_{-2}, \\ d(x_3) &= (b-c)x_3, & d(x_{-3}) &= -(b-c)x_{-3}. \end{aligned} \tag{6.3}$$

The same calculations show that any transformation defined by (6.3) with b, c arbitrary scalars is a derivation of $\mathbb{O}_q$.

Now suppose that d is an arbitrary derivation of $\mathbb{O}_q$, and let $d(e_1) = a_0e_1 + a'_0e_2 + \sum_{j=-3,j\neq 0}^{3} a_jx_j$ and $d(e_2) = b_0e_1 + b'_0e_2 + \sum_{j=-3,j\neq 0}^{3} b_jx_j$. Computing $d(e_1) \cdot e_1 + e_1 \cdot d(e_1) = d(e_1)$, we obtain

$$2a_0e_1 + q^{-2}a_1x_1 + a_2x_2 + a_3x_3 + q^{-2}a_{-1}x_{-1} + a_{-2}x_{-2} + a_{-3}x_{-3} = d(e_1),$$

which shows that $a_0 = a'_0 = a_{-1} = a_1 = 0$. A similar calculation with the relation $d(e_2^2) = d(e_2)$ gives $b_0 = b'_0 = b_{-1} = b_1 = 0$. From $d(e_1) \cdot e_2 + e_1 \cdot d(e_2) = 0$ it follows that $b_2 = -a_2$ and $b_3 = -a_3$, and from $d(e_2) \cdot e_1 + e_2 \cdot d(e_1) = 0$, the relations $b_{-2} = -a_{-2}$, $b_{-3} = -a_{-3}$ can be determined.

Now $e_1 \cdot d(x_1) = -d(e_1) \cdot x_1$ shows that the coefficients of x_2 and x_3 in $d(x_1)$ are $q^{\frac{3}{2}}a_{-3}$ and $q^{\frac{3}{2}}a_{-2}$ respectively; while $d(x_1) \cdot e_2 = -x_1 \cdot d(e_2)$ says that those coefficients are $-q^{-\frac{3}{2}}b_{-3} = q^{-\frac{3}{2}}a_{-3}$ and $-q^{-\frac{3}{2}}b_{-2} = q^{-\frac{3}{2}}a_{-2}$. Thus, when q is not a cube root of unity, we obtain that $a_{-2} = 0 = a_{-3}$. Likewise the relations $a_2 = 0 = a_3$ can be found from analyzing $d(x_{-1}) \cdot e_1 = -x_{-1}d(e_1)$ and $e_2 \cdot d(x_{-1}) = -d(e_2) \cdot x_{-1}$. Consequently, when q is not a cube root of unity, every derivation of $\mathbb{O}_q$ must annihilate e_1 and e_2, and so must be as in (6.3) for suitable scalars b, c. To summarize, we have

Theorem 6.4. *The derivation algebra of the quantum octonion algebra* $\mathbb{O}_q = (V, \cdot)$ *is two-dimensional. Every derivation* d *of* $\mathbb{O}_q$ *is given by (6.3) for some scalars* $b, c \in \mathbf{K}$.

Simplicity of the quantum octonions. The idempotents also provide a proof of the following.

Theorem 6.5. *The quantum octonion algebra* $\mathbb{O}_q = (V, \cdot)$ *is simple.*

Proof. Let I be a nonzero ideal of $\mathbb{O}_q$. Since $e_1 I \subseteq I$ and $I e_1 \subseteq I$, it follows that

$$I = \bigoplus_{\gamma,\delta} I_{\gamma,\delta},$$

where $I_{\gamma,\delta} = \{x \in I \mid e_1 \cdot x = \gamma x \text{ and } x \cdot e_1 = \delta x\}$. Assume first that $x_i \in I$ for some $i = \{\pm 1, \pm 2, \pm 3, \pm 4\}$. If we multiply x_i by x_{-i} when $i = \pm 1, \pm 2$, or ± 3, we see that e_1 or e_2 belongs to I. Therefore, either $e_1 \mathbb{O}_q + \mathbb{O}_q e_1 \subseteq I$ or $e_2 \mathbb{O}_q + \mathbb{O}_q e_2 \subseteq I$, and as a consequence, $I = \mathbb{O}_q$. It remains to consider the possibility that some nonzero linear combination $a x_2 + b x_3$ or $a x_{-2} + b x_{-3}$ is in I. Multiplying by one of the elements x_2, x_3, x_{-2}, or x_{-3}, we obtain that x_1 or x_{-1} is in I, and that implies $I = \mathbb{O}_q$ as before. Consequently, $\mathbb{O}_q$ is simple. □

7. Quantum Quaternions

The subalgebra $\mathbb{H}_q$ of $\mathbb{O}_q$ generated by $x = x_2$ and $y = x_{-2}$ is 4-dimensional and has x, y, e_1, e_2 as a basis, where $e_1 = q^{-\frac{1}{2}} x_4$ and $e_2 = -q^{\frac{1}{2}} x_{-4}$ are the orthogonal idempotents of Section 6. The sum $e = e_1 + e_2$ is the identity element of $\mathbb{H}_q$, and multiplication in this algebra is given by Table 7.1. It is easy to see that at $q = 1$ this algebra is isomorphic to the algebra $M_2(\mathbf{K})$ of 2×2 matrices over $\mathbf{K}$ via the mapping $e_1 \mapsto E_{1,1}$, $e_2 \mapsto E_{2,2}$, $x \mapsto E_{1,2}$, $y \mapsto E_{2,1}$, which sends basis elements to standard matrix units. Since the split quaternion algebra is isomorphic to $M_2(\mathbf{K})$, this algebra reduces to the quaternions at $q = 1$. Another way to see this is to observe that the bilinear form on $\mathbb{O}_q$, when restricted to $\mathbb{H}_q$, satisfies (3.6) at $q = 1$. Thus, at $q = 1$ we have a 4-dimensional unital algebra with a nondegenerate bilinear form admitting composition, and so it must be the quaternions.

Table 7.1

$\cdot$	e_1	e_2	x	y
e_1	e_1	0	x	0
e_2	0	e_2	0	y
x	0	x	0	$q^{-1}e_1$
y	y	0	qe_2	0

The algebra $\mathbb{H}_q$ is neither associative nor flexible if $q \neq \pm 1$, since

$$(x \cdot y) \cdot x = q^{-1} x \neq q x = x \cdot (y \cdot x).$$

However, it is *Lie admissible*. Indeed, if $h = q^{-1} e_1 - q e_2$, then

$$[h, x] = (q + q^{-1}) x, \quad [h, y] = -(q + q^{-1}) y, \quad \text{and} \quad [x, y] = h.$$

Therefore, the elements x, $\dfrac{2}{q+q^{-1}}y$, $\dfrac{2}{q+q^{-1}}h$ determine a standard basis for sl_2. Since $e = e_1 + e_2$ is the identity, we see that $\mathbb{H}_q$ is the Lie algebra gl_2 under the commutator product $[a,b] = ab - ba$.

The subalgebra generated by x_3 and x_{-3} is isomorphic to the algebra $\mathbb{H}_q$. However, that is not true if we consider the subalgebra $\mathbb{H}'_q$ generated by x_1 and x_{-1}. Let $u = x_{-1}$ and $v = x_1$. Then multiplication in $\mathbb{H}'_q$ is given by Table 7.2.

Table 7.2

$\cdot$	e_1	e_2	u	v
e_1	e_1	0	$q^{-2}u$	0
e_2	0	e_2	0	q^2v
u	0	q^2u	0	q^2e_1
v	$q^{-2}v$	0	$q^{-2}e_2$	0

There is no identity element in $\mathbb{H}'_q$. But at $q = 1$, the element $e_1 + e_2$ becomes an identity element, and the algebra is isomorphic to $M_2(\mathbf{K})$. The bilinear form at $q = 1$ has the composition property, so the algebra $\mathbb{H}'_q$ too might rightfully be termed a quantum quaternion algebra. The elements $h' = q^2e_1 - q^{-2}e_2, u, v$ determine a standard basis for sl_2:

$$[h', u] = 2u, \qquad [h', v] = -2v, \qquad [u, v] = h'.$$

Since $[q^2e_1 + q^{-2}e_2, h'] = [q^2e_1 + q^{-2}e_2, u] = [q^2e_1 + q^{-2}e_2, v] = 0$, the algebra $\mathbb{H}'_q$ under the commutator product is isomorphic to gl_2. It is neither associative nor flexible, because

$$\big((e_1 + v)\cdot e_1\big)\cdot(e_1 + v) = e_1 + q^{-4}v \neq e_1 + q^{-2}v = (e_1 + v)\cdot\big(e_1\cdot(e_1 + v)\big).$$

8. Connections with $U_q(\mathrm{B}_3)$ and $U_q(\mathrm{G}_2)$

In this section we assume that the field $\mathbf{K}$ has characteristic zero. We consider the subalgebras of $U_q(\mathrm{D}_4)$ that are described as the fixed points of graph automorphisms by

$$\begin{aligned} U_{\zeta,\eta} &= \{a \in U_q(\mathrm{D}_4) \mid \zeta(a) = a = \eta(a)\},\\ U_\delta &= \{a \in U_q(\mathrm{D}_4) \mid \delta(a) = a\}, \end{aligned} \tag{8.1}$$

where $\delta = (3\,4) = \zeta\eta\zeta$. Then $U_\delta \supseteq U_{\zeta,\eta}$. In the non-quantum case, the enveloping algebras $U(\mathrm{G}_2) \subseteq U_{\zeta,\eta} \subseteq U(\mathrm{D}_4)$ and $U(\mathrm{B}_3) \subseteq U_\delta \subseteq U(\mathrm{D}_4)$. We consider the structure of V, V_ζ, V_η as modules for the subalgebras given in (8.1).

Now for any $a \in U_\delta$, we have that

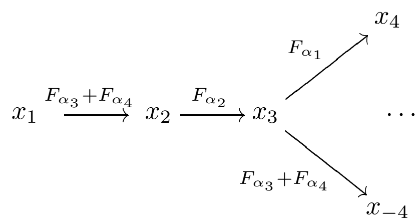
$$\jmath(ax) = \jmath(\delta(a)x) = a\jmath(x)$$

for all $x \in V$, so that any $a \in U_\delta$ commutes with $\jmath$. In particular, the eigenspaces of $\jmath$ are U_δ-invariant. But the eigenspaces of $\jmath$ are 1-dimensional (spanned by $x_4 - x_{-4}$), and 7-dimensional (spanned by $\{x_i \mid i = \pm1, \pm2, \pm3\} \cup \{x_4 + x_{-4}\}$). Consider the 7-dimensional space under the action of $U_{\zeta,\eta}$. Since $F_{\alpha_2}, F_{\alpha_1} + F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_2}, E_{\alpha_1} + E_{\alpha_3} + E_{\alpha_4} \in U_{\zeta,\eta}$, it is easy to check that the 7-dimensional space is an irreducible $U_{\zeta,\eta}$-module, hence an irreducible U_δ-module as well. The action of $U_{\zeta,\eta}$ on V_ζ and V_η is the same as its action on V. As a result we have

Proposition 8.2. *The modules $V, V_\zeta \cong V_-$, and $V_\eta \cong V_+$ are isomorphic as modules for the subalgebra $U_{\zeta,\eta}$ in (8.1). They decompose into the sum of irreducible $U_{\zeta,\eta}$-modules of dimensions 1 and 7, which are the eigenspaces of $\jmath$.*

For $q = 1$, Proposition 8.2 amounts to saying that V, V_ζ, V_η are isomorphic $U(\mathrm{G}_2)$-modules which decompose into irreducible modules of dimension 1 and 7.

The elements $F_{\alpha_1}, F_{\alpha_2}, F_{\alpha_3} + F_{\alpha_4}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_3} + E_{\alpha_4}$ belong to U_δ, so if we regard V_ζ or V_η as a module for U_δ, then we obtain a diagram



which is gotten by superimposing the two diagrams of V_ζ and V_η. Using that, it is easy to argue that both these modules remain irreducible for U_δ. Moreover, for any $a \in U_\delta$, we have

$$\begin{aligned}\jmath(\zeta(a)x) &= \jmath((\zeta\delta)(a)x) = \jmath((\eta\zeta)(a)x)\\ &= \jmath(\theta^2(a)x) = \eta(a)\jmath(x),\end{aligned}$$

where $\theta = (1\,4\,3)$. Consequently, $V_\zeta \cong V_\eta$ as U_δ-modules.

At $q = 1$ these computations show that V_ζ and V_η (the two spin modules of $U(\mathrm{D}_4)$) remain irreducible for $U(\mathrm{B}_3)$, and in fact they are the spin representation of that algebra. To summarize, we have

Proposition 8.3. *When regarded as modules for the subalgebra U_δ of $U_q(\mathrm{D}_4)$, $V_\zeta \cong V_-$ and $V_\eta \cong V_+$ are irreducible and isomorphic.*

In light of these propositions it is natural to expect that these fixed point subalgebras of the graph automorphisms might be the quantum groups $U_q(\mathrm{G}_2)$ and $U_q(\mathrm{B}_3)$. However, that is not true. In fact the next "nonembedding" result shows that there are no Hopf algebra homomorphisms of them into $U_q(\mathrm{D}_4)$ except for the trivial ones given by the counit.

Proposition 8.4. *When q is not a root of unity, then any Hopf algebra homomorphisms*

$$U_q(\mathrm{G}_2) \to U_q(\mathrm{B}_3), \quad U_q(\mathrm{B}_3) \to U_q(\mathrm{D}_4), \quad U_q(\mathrm{G}_2) \to U_q(\mathrm{D}_4)$$

are trivial ($a \mapsto \epsilon(a)1$ for all a).

The method we use to establish this result is due originally to Hayashi [H]. We begin with a little background before presenting the proof.

Suppose $U_q(\mathfrak{g})$ is a quantum group and (X, π), $\pi : U_q(\mathfrak{g}) \to \mathrm{End}(X)$, is a representation of $U_q(\mathfrak{g})$. This determines another representation (X, π'), $\pi' = \pi \circ S^2$, of $U_q(\mathfrak{g})$, where S is the antipode. But it is well-known [Jn, p. 56] that S^2 has the following expression:

$$S^2(a) = K_{2\rho}^{-1} a K_{2\rho} \qquad \text{for all } a \in U_q(\mathfrak{g}),$$

where $\rho = \frac{1}{2}\sum_{\gamma>0}\gamma = \sum_{\alpha\in\Pi} c_\alpha \alpha$ (the half-sum of the positive roots of $\mathfrak{g}$) and $K_{2\rho} = \prod_{\alpha\in\Pi} K_\alpha^{c_\alpha}$ (where Π is the set of simple roots of $\mathfrak{g}$). The two representations (X,π) and (X,π') are in fact isomorphic by the following map:

$$\phi : (X,\pi) \to (X,\pi'), \qquad x \mapsto \pi(K_{2\rho}^{-1})x,$$

because

$$\begin{aligned}\phi(\pi(a)x) &= \pi(K_{2\rho}^{-1})\pi(a)x = \pi(K_{2\rho}^{-1}a)x = \pi(S^2(a)K_{2\rho}^{-1})x \\ &= \pi(S^2(a))\pi(K_{2\rho}^{-1})x = \pi'(a)\phi(x) \quad \text{for all } a \in U_q(\mathfrak{g}).\end{aligned}$$

Examples 8.5. Let $\mathfrak{g} = \mathrm{D}_4$ and let $X = V, V_-, V_+$ (the natural or spin representations of $U_q(\mathrm{D}_4)$). Here $\rho = 3\alpha_1 + 5\alpha_2 + 3\alpha_3 + 3\alpha_4$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-6}K_{\alpha_2}^{-10}K_{\alpha_3}^{-6}K_{\alpha_4}^{-6}$. Relative to the basis $\{x_1, x_2, x_3, x_4, x_{-4}, x_{-3}, x_{-2}, x_{-1}\}$ we have, as before,

$$\phi_{\mathrm{D}_4} = \phi_{\mathrm{D}_4}^- = \phi_{\mathrm{D}_4}^+ = \mathrm{diag}\{q^{-6}, q^{-4}, q^{-2}, 1, 1, q^2, q^4, q^6\}. \tag{8.6}$$

Next let $\mathfrak{g} = \mathrm{G}_2$ and let X be the 7-dimensional irreducible $U_q(\mathrm{G}_2)$-representation. Here $\rho = 5\alpha_1 + 3\alpha_2$, so $K_{2\rho}^{-1} = K_{\alpha_1}^{-10}K_{\alpha_2}^{-6}$. With respect to the basis (see [Jn], Chap. 5)

$$\{x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_1}, x_0, x_{-\alpha_1}, x_{-(\alpha_1+\alpha_2)}, x_{-(2\alpha_1+\alpha_2)}\}, \tag{8.7}$$

we have

$$\phi_{\mathrm{G}_2} = \mathrm{diag}\{q^{-10}, q^{-8}, q^{-2}, 1, q^2, q^8, q^{10}\}. \tag{8.8}$$

Finally, let X be the 7-dimensional natural representation or the 8-dimensional spin representation of $U_q(\mathrm{B}_3)$. In this case $2\rho = 5\alpha_1 + 8\alpha_2 + 9\alpha_3$, so that $K_{2\rho}^{-1} = K_{\alpha_1}^{-5}K_{\alpha_2}^{-8}K_{\alpha_3}^{-9}$. If we assume as in [Jn] that a short root satisfies $(\alpha,\alpha) = 2$, then setting $(\epsilon_i, \epsilon_i) = 2$ for $i = 1, 2, 3$, we have that the matrix of the bilinear form relative to the basis $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$, $\alpha_3 = \epsilon_3$ of simple roots is

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

From this we can compute that $(-2\rho, \epsilon_i) = -10, -6, -2$ for $i = 1, 2, 3$, respectively. Since we know, for any weight vector x_μ, that $K_\alpha x_\mu = q^{(\mu,\alpha)}x_\mu$, we obtain that, relative to the basis

$$\{x_{\epsilon_1}, x_{\epsilon_2}, x_{\epsilon_3}, x_0, x_{-\epsilon_3}, x_{-\epsilon_2}, x_{-\epsilon_1}\}, \tag{8.9}$$

for the natural representation,

$$\phi_{\mathrm{B}_3} = \mathrm{diag}\{q^{-10}, q^{-6}, q^{-2}, 1, q^2, q^6, q^{10}\}. \tag{8.10}$$

The spin representation of $U_q(\mathrm{B}_3)$ has a basis

$$\{x_{\xi_1}, x_{\xi_2}, x_{\xi_3}, x_{\xi_4}, x_{-\xi_4}, x_{-\xi_3}, x_{-\xi_2}, x_{-\xi_1}\},$$

where $\xi_1 = (1/2)(\epsilon_1 + \epsilon_2 + \epsilon_3)$, $\xi_2 = (1/2)(\epsilon_1 + \epsilon_2 - \epsilon_3)$, $\xi_3 = (1/2)(\epsilon_1 - \epsilon_2 + \epsilon_3)$, and $\xi_4 = (1/2)(-\epsilon_1 + \epsilon_2 + \epsilon_3)$. The corresponding matrix of $\phi_{\mathrm{B}_3}^+$ is given by

$$\phi_{\mathrm{B}_3}^+ = \mathrm{diag}\{q^{-9}, q^{-7}, q^{-3}, q^{-1}, q, q^3, q^7, q^9\}.$$

Now suppose that $f : U_q(\mathfrak{g}_1) \to U_q(\mathfrak{g}_2)$ is a Hopf algebra homomorphism between two quantum groups. Consider representations (X,π) and (X,π') for $U_q(\mathfrak{g}_2)$ as above. Then $(X, \pi \circ f)$, $(X, \pi' \circ f)$ are representations for $U_q(\mathfrak{g}_1)$. Since $\pi' \circ f = \pi \circ S^2 \circ f = \pi \circ f \circ S^2$, it follows that $\pi' \circ f = (\pi \circ f)'$. Moreover, the map

$$\phi_{\mathfrak{g}_2} : (X, \pi) \to (X, \pi')$$

satisfies

$$\phi_{\mathfrak{g}_2}((\pi \circ f)(a)x) = \pi'(f(a))\phi_{\mathfrak{g}_2}(x) = (\pi \circ f)'(a)\phi_{\mathfrak{g}_2}(x)$$

for all $a \in U_q(\mathfrak{g}_1)$ and $x \in X$, so it is an isomorphism of $U_q(\mathfrak{g}_1)$-modules: $\phi_{\mathfrak{g}_2} : (X, \pi \circ f) \to (X, (\pi \circ f)')$.

Proof of Proposition 8.4. Suppose $f : U_q(\mathrm{G}_2) \to U_q(\mathrm{B}_3)$ is a Hopf algebra homomorphism. Consider the natural representation $Y = (X, \pi)$ of $U_q(\mathrm{B}_3)$, and the corresponding representation

$$\phi_{\mathrm{B}_3} : (X, \pi \circ f) \to (X, (\pi \circ f)') \tag{8.11}$$

of $U_q(\mathrm{G}_2)$-modules. Either these are the 7-dimensional irreducible $U_q(\mathrm{G}_2)$-module or they are the sum of 7 trivial 1-dimensional modules. In the first case ϕ_{B_3} is a multiple of ϕ_{G_2}, so the eigenvalues of ϕ_{B_3} are $\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\}$. This is impossible since q is not a root of unity, so it must be that $\pi \circ f$ is trivial.

Analogously, consider the spin representation $Y^+ = (X^+, \pi^+)$ of $U_q(\mathrm{B}_3)$. Then as a $U_q(\mathrm{G}_2)$-module, such a representation should be the sum of representations of dimension 1 or 7. Reasoning as in (8.11), we see that if a 7-dimensional $U_q(\mathrm{G}_2)$-module occurs, then $\phi^+_{\mathrm{B}_3}$ restricted to the 7-dimensional submodule should be proportional to ϕ_{G_2}. This would mean that

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \subseteq \mathrm{diag}\{q^{\pm 9}, q^{\pm 7}, q^{\pm 3}, q^{\pm 1}\}$$

for some scalar λ, which is impossible. Thus $\pi^+ \circ f$ is trivial.

We have seen that the natural and spin representations for $U_q(\mathrm{B}_3)$ are trivial modules for $f(U_q(\mathrm{G}_2))$. Thus, $f(a) - \epsilon(f(a))1$ is in the annihilator of those representations. Since any finite-dimensional irreducible $U_q(\mathrm{B}_3)$-representation of type I can be obtained from the tensor product of those, and since f is a Hopf homomorphism, it follows that $f(U_q(\mathrm{G}_2))$ acts trivially on any finite-dimensional irreducible $U_q(\mathrm{B}_3)$-representation of type I. However, $\bigcap \mathrm{ann}(V) = (0)$ for all such representations (see [Jn, Prop. 5.11]). Consequently, $f(a) = \epsilon(f(a))1$ for all $a \in U_q(\mathrm{G}_2)$.

The other two cases in Proposition 8.4 are similar. For a Hopf homomorphism $f : U_q(\mathrm{G}_2) \to U_q(\mathrm{D}_4)$, it boils down to the fact that any finite-dimensional irreducible $U_q(\mathrm{D}_4)$-representation occurs in tensor products of copies of the natural or spin representations and

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 8}, \lambda q^{\pm 2}, \lambda\} \not\subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}.$$

In the final case, $f : U_q(\mathrm{B}_3) \to U_q(\mathrm{D}_4)$, the argument amounts to the impossibility of

$$\{\lambda q^{\pm 10}, \lambda q^{\pm 6}, \lambda q^{\pm 2}, \lambda\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\},$$

and

$$\{\lambda q^{\pm 9}, \lambda q^{\pm 7}, \lambda q^{\pm 3}, \lambda q^{\pm 1}\} \subseteq \{q^{\pm 6}, q^{\pm 4}, q^{\pm 2}, 1\}. \qquad \square$$

9. Quantum Octonions and Quantum Clifford Algebras

In this section we investigate the representation of the quantum Clifford algebra $C_q(8)$ on $\mathbb{O}_q \oplus \mathbb{O}_q$. Our main reference for the quantum Clifford algebra is [DF], but we need to make several adjustments to make the results of [DF] conform to our work here. It is convenient to begin with the general setting of the quantum group of D_n and then to specialize to D_4.

Explicit form of $\check{R}_{V,V}$ for $U_q(\mathrm{D}_n)$. The natural representation V of the quantum group $U_q(\mathrm{D}_n)$ has a basis

$$\{x_1, x_2, \dots, x_{2n}\} = \{x_{\epsilon_1}, \dots, x_{\epsilon_n}, x_{-\epsilon_n}, \dots, x_{-\epsilon_1}\}, \tag{9.1}$$

where $E_\alpha, F_\alpha, K_\alpha$ act as in (2.2). A straightforward computation shows that relative to the basis $\{x_1, x_2, \dots, x_{2n}\}$, the matrix of $\check{R} = \check{R}_{V,V}$ is given by

$$\begin{aligned} \check{R} = q^{-1}\sum_{i=1}^{2n} E_{i,i}\otimes E_{i,i} + \sum_{i\neq j,j'} E_{j,i}\otimes E_{i,j} + (q^{-1}-q)\sum_{j>i} E_{j,j}\otimes E_{i,i} \\ + q\sum_{i=1}^{2n} E_{i',i}\otimes E_{i,i'} + (q-q^{-1})\sum_{i<j}(-q)^{\rho_{j'}-\rho_{i'}} E_{i',j}\otimes E_{i,j'}, \end{aligned} \tag{9.2}$$

where $i' = 2n+1-i$ and $\rho = (\rho_1, \dots, \rho_{2n}) = (n-1, n-2, \dots, 1, 0, 0, -1, \dots, 2-n, 1-n)$. The projection $(\ ,\ ) : V\otimes V \to K$ is given by

$$(x_i, x_j) = \delta_{i,j'}(-1)^{n-1}(-q)^{\rho_{i'}}.$$

If we modify the basis slightly by defining

$$e_i = (-1)^{n-1+\rho_{i'}} x_i, \qquad e_{i'} = x_{i'} \tag{9.3}$$

for $i = 1, \dots, n$, then we obtain the usual expression for $\check{R}$:

$$\begin{aligned} \check{R} = q^{-1}\sum_{i=1}^{2n} E_{i,i}\otimes E_{i,i} + \sum_{i\neq j,j'} E_{j,i}\otimes E_{i,j} + (q^{-1}-q)\sum_{j>i} E_{j,j}\otimes E_{i,i} \\ + q\sum_{i=1}^{2n} E_{i',i}\otimes E_{i,i'} + (q-q^{-1})\sum_{i<j} q^{(\rho_{j'}-\rho_{i'})} E_{i',j}\otimes E_{i,j'}, \end{aligned} \tag{9.4}$$

and the bilinear form is given by

$$(e_i, e_j) = \delta_{i,j'} q^{\rho_{i'}}.$$

The quantum Clifford algebra. In [DF], Ding and Frenkel define the quantum Clifford algebra $C_q(2n)$ as the quotient of the tensor algebra $T(V)$ of V by the ideal generated by $\{v\otimes w + q\check{R}'_{2,1}(v\otimes w) - (w,v)1 \mid v, w \in V\}$, where $\check{R}'$ is $\check{R}$ with q and q^{-1} interchanged, and $\check{R}'_{2,1} = \sigma\circ\check{R}'\circ\sigma$, where $\sigma(v\otimes w) = w\otimes v$. This quantum Clifford algebra can be thought of as a unital associative algebra generated by $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$, with relations

$$\begin{aligned} &\psi_i\psi_j = -q^{-1}\psi_j\psi_i, \qquad \psi_i^*\psi_j^* = -q\psi_j^*\psi_i^* \quad (i>j), \\ &\psi_i\psi_i = 0 = \psi_i^*\psi_i^*, \qquad \psi_i\psi_j^* = -q^{-1}\psi_j^*\psi_i \quad (i\neq j), \\ &\psi_i\psi_i^* + \psi_i^*\psi_i = (q^{-2}-1)\sum_{i<j}\psi_j\psi_j^* + 1. \end{aligned} \tag{9.5}$$

If we make the change of notation $Y_i = \psi^*_{n-i}, Y^*_i = \psi_{n-i}$, $i = 1, \dots, n$, then the relations in (9.5) become

$$\begin{aligned} &Y_iY_j = -q^{-1}Y_jY_i, \qquad Y^*_iY^*_j = -qY^*_jY^*_i \quad (i > j),\\ &Y_iY_i = 0 = Y^*_iY^*_i, \qquad Y_iY^*_j = -qY^*_jY_i \quad (i \neq j),\\ &Y_iY^*_i + Y^*_iY_i = (q^{-2} - 1)\sum_{i>j} Y_jY^*_j + 1. \end{aligned} \tag{9.6}$$

In order to construct a representation of $C_q(8)$ on $\mathbb{O}_q \oplus \mathbb{O}_q$ we will need the following proposition:

Proposition 9.7. *Let I be the ideal of the tensor algebra $T(V)$ generated by $\{v \otimes w + q^{-1}\check{R}_{2,1}(v \otimes w) - (w, v)1 \mid v, w \in V\}$, where $\check{R}_{2,1} = \sigma \circ \check{R} \circ \sigma$. Then $C_q(2n) \cong T(V)/I$.*

Proof. Consider the following basis of V:

$$\{y_i = q^{\rho_i}e_i, \ \ y^*_i = e_{i'} \mid i = 1, \dots, n\}, \tag{9.8}$$

which satisfies

$$(y_i, y^*_j) = \delta_{i,j}, \qquad (y^*_j, y_i) = \delta_{i,j}q^{2\rho_i}. \tag{9.9}$$

Let $g : V \otimes V \to I$ be defined by

$$g(v \otimes w) = v \otimes w + q^{-1}\check{R}_{2,1}(v \otimes w) - (w, v)1. \tag{9.10}$$

From the explicit form of $\check{R}$ on the basis $\{e_i\}$, one can check that the following hold:

$$\begin{aligned} &g(y_i \otimes y_j) = y_i \otimes y_j + q^{-1}y_j \otimes y_i \quad (i > j),\\ &g(y^*_j \otimes y^*_i) = y^*_j \otimes y^*_i + q^{-1}y^*_i \otimes y^*_j \quad (i > j),\\ &g(y_i \otimes y_i) = (1 + q^{-2})y_i \otimes y_i, \qquad g(y^*_i \otimes y^*_i) = (1 + q^{-2})y^*_i \otimes y^*_i,\\ &g(y^*_j \otimes y_i) = y^*_j \otimes y_i + q^{-1}y_i \otimes y^*_j \quad (i \neq j),\\ &g(y^*_i \otimes y_i) = y^*_i \otimes y_i + y_i \otimes y^*_i + (1 - q^{-2})\sum_{i>j} y_j \otimes y^*_j - 1. \end{aligned} \tag{9.11}$$

These give the relations in (9.6) in $T(V)/I$. However, we also have the relations

$$\begin{aligned} &g(y_j \otimes y_i) = q^{-1}y_i \otimes y_j + q^{-2}y_j \otimes y_i \quad (i > j),\\ &g(y^*_i \otimes y^*_j) = q^{-1}y^*_j \otimes y^*_i + q^{-2}y^*_i \otimes y^*_j \quad (i > j),\\ &g(y_i \otimes y^*_j) = q^{-1}y^*_j \otimes y_i + q^{-2}y_i \otimes y^*_j \quad (i \neq j),\\ &g(y_i \otimes y^*_i) = y^*_i \otimes y_i + q^{-2}y_i \otimes y^*_i\\ &\qquad\qquad + (1 - q^{-2})q^{2\rho_i}\Big(\sum_{j=1}^{n} y_j \otimes y^*_j + \sum_{j>i} q^{-2\rho_j}y^*_j \otimes y_j\Big) - q^{2\rho_i}. \end{aligned} \tag{9.12}$$

The first three identities don't say anything new. So to establish the proposition, it suffices to verify that the relation $g(y_i \otimes y^*_i) = 0$ holds in $C_q(2n)$, that is,

Lemma 9.13. *In $C_q(2n)$ we have*

$$q^{-2}Y_iY^*_i + Y^*_iY_i + (1 - q^{-2})q^{2\rho_i}\Big(\sum_{j=1}^{n} Y_jY^*_j + \sum_{j>i} q^{-2\rho_j}Y^*_jY_j\Big) = q^{2\rho_i}.$$

Proof. As a shorthand notation we set $a_i = Y_iY_i^*$, $a_i' = Y_i^*Y_i$, and $b_i = a_i + a_i'$ for $i = 1, \dots, n$. From the relations in $C_q(2n)$ we know that $b_i = (q^{-2}-1)\sum_{j<i} a_j + 1$, so that $b_{i+1} = (q^{-2}-1)a_i + b_i$ or

$$a_{i+1} + a_{i+1}' = q^{-2}a_i + a_i'. \tag{9.14}$$

Letting $c_i = q^{-2}a_i + a_i'$, we have

$$c_i - q^2c_{i+1} = q^{-2}a_i + a_i' - a_{i+1} - q^2a_{i+1}' \overset{1}{=} (1-q^2)a_{i+1}',$$

where (1) follows from (9.14). Therefore,

$$\begin{aligned} c_i &= (1-q^2)a_{i+1}' + q^2(1-q^2)a_{i+2}' + \cdots + q^{2(n-i-1)}(1-q^2)a_n' + q^{2(n-i)}c_n \\ &= (q^{-2}-1)q^{2\rho_i}\Big(\sum_{j>i} q^{-2\rho_j}a_j'\Big) + q^{2\rho_i}c_n. \end{aligned} \tag{9.15}$$

On the other hand,

$$\begin{aligned} c_n &= q^{-2}a_n + a_n' = (q^{-2}-1)a_n + a_n + a_n' = (q^{-2}-1)a_n + b_n \\ &= (q^{-2}-1)a_n + (q^{-2}-1)\Big(\sum_{j<n} a_j\Big) + 1 = (q^{-2}-1)\Big(\sum_{j\le n} a_j\Big) + 1. \end{aligned} \tag{9.16}$$

The left-hand side of the identity in the statement of the lemma is

$$\begin{aligned} c_i + (1-q^{-2})q^{-2\rho_i}&\Big(\sum_{j=1}^{n} a_j\Big) + (1-q^{-2})q^{2\rho_i}\Big(\sum_{j>i} q^{-2\rho_j}a_j'\Big) \\ &\overset{1}{=} c_i + q^{2\rho_i}(1-c_n) + (1-q^{-2})q^{2\rho_i}\Big(\sum_{j>i} q^{-2\rho_j}a_j'\Big) \\ &\overset{2}{=} q^{2\rho_i} \quad \text{(the right side)}, \end{aligned}$$

where (1) is (9.16) and (2) is (9.15). □

The $U_q(\mathbf{D}_4)$ case. Let us return now to our quantum octonions. Consider the two "intertwining operators" Φ^0 and Φ^1 which are the $U_q(\mathrm{D}_4)$-module homomorphisms

$$\begin{aligned} \Phi^0 &: V_\eta \otimes V \overset{\jmath\otimes \mathrm{id}}{\to} V_{\eta\circ\zeta} \otimes V \overset{\jmath\circ p\circ(\jmath\otimes\jmath)}{\to} V_\zeta, \\ \Phi^1 &: V_\zeta \otimes V \overset{\jmath\otimes \mathrm{id}}{\to} V_{\zeta\circ\eta} \otimes V \overset{p}{\to} V_\eta. \end{aligned}$$

We scale these mappings by setting $\Psi^0 = q^{-\frac{3}{2}}\Phi^0$ and $\Psi^1 = q^{-\frac{3}{2}}\Phi^1$. Let $\mathfrak{S}_q = (\mathrm{id}{+}q^{-1}\check{R})$, where $\check{R} = \check{R}_{V,V}$, which we know maps $V\otimes V$ onto $\mathcal{S}^2(V)$.

Lemma 9.17 (Compare [DF, Prop. 3.1.2])**.**

$$\begin{aligned} \Psi^1 \circ (\Psi^0 \otimes \mathrm{id}) \circ (\mathrm{id}\otimes \mathfrak{S}_q) &= \mathrm{id} \otimes (\ ,\ ), \\ \Psi^0 \circ (\Psi^1 \otimes \mathrm{id}) \circ (\mathrm{id}\otimes \mathfrak{S}_q) &= \mathrm{id} \otimes (\ ,\ ). \end{aligned}$$

Proof. It is enough to check the values on $x_1\otimes x_1\otimes x_{-1}$:

$$\begin{aligned} \Psi^1 \circ (\Psi^0 &\otimes \mathrm{id}) \circ (\mathrm{id}\otimes \mathfrak{S}_q)(x_1\otimes x_1\otimes x_{-1}) \\ &= \Psi^1 \circ (\Psi^0 \otimes \mathrm{id})(x_1 \otimes (x_1\otimes x_{-1} + x_{-1}\otimes x_1)) \\ &= q^{-\frac{3}{2}}\Psi^1(q^{-\frac{3}{2}}x_4\otimes x_1) = q^{-3}x_1. \end{aligned}$$

But $(x_1, x_{-1}) = (q^3+q^{-3})(x_1|x_{-1}) = q^{-3}$, so both maps agree on $x_1\otimes x_1\otimes x_{-1}$. The computation for the other map is left as an exercise. □

Now for any $v \in V$ define

$$\begin{aligned} &\Psi(v) : V_\zeta \oplus V_\eta \to V_\zeta \oplus V_\eta, \\ &\quad \Psi(v)(x+y) = \Psi^0(y \otimes v) + \Psi^1(x \otimes v). \end{aligned} \tag{9.18}$$

The relations in the previous lemma say that

$$\Psi \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{S}_q) = \mathrm{id} \otimes (\ ,\ ),$$

so for any basis elements $e_i, e_j \in V$ and $x \in V_\zeta \oplus V_\eta$, we have

$$\begin{aligned} (e_j, e_i)x &= \Psi \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{S}_q)(x \otimes e_j \otimes e_i) \\ &= \Psi \circ (\Psi \otimes \mathrm{id})(x \otimes (e_j \otimes e_i + q^{-1}\check{R}(e_j \otimes e_i))) \\ &= \Psi(e_i)\Psi(e_j)(x) + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} \Psi(e_\ell)\Psi(e_k)(x), \end{aligned}$$

where $(\check{R} \circ \sigma)_{i,j}^{k,\ell}$ is the coordinate matrix of $\check{R} \circ \sigma$ in the basis $\{e_i \otimes e_j\}$. So $\Psi(e_i)$ and $\Psi(e_j)$ satisfy

$$(e_j, e_i)\mathrm{id} = \Psi(e_i)\Psi(e_j) + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} \Psi(e_\ell)\Psi(e_k).$$

Notice now that

$$\begin{aligned} e_i \otimes e_j + q^{-1}\check{R}_{2,1}(e_i \otimes e_j) &= e_i \otimes e_j + q^{-1}\sigma \circ \check{R} \circ \sigma(e_i \otimes e_j) \\ &= e_i \otimes e_j + q^{-1} \sum_{k,\ell} (\check{R} \circ \sigma)_{i,j}^{k,\ell} e_\ell \otimes e_k, \end{aligned}$$

so the operators $\Psi(v)$ satisfy the relations of the quantum Clifford algebra. This shows that the map

$$\begin{aligned} &\Psi : V \to \mathrm{End}(V_\zeta \oplus V_\eta), \\ &\quad v \mapsto \Psi(v) : x + y \to q^{-3}\Big(\jmath(y \cdot \jmath(v)) + \jmath(x) \cdot v\Big), \end{aligned} \tag{9.19}$$

extends to a representation $\Psi : C_q(8) \to \mathrm{End}(V_\zeta \oplus V_\eta) = \mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$.

In [DF], Ding and Frenkel show that $C_q(2n)$ is isomorphic to the classical Clifford algebra, so $C_q(8)$ is a central simple associative algebra of dimension 2^8. The same is true of $\mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$. As a result we have

Proposition 9.20. *The map* $\Psi : C_q(8) \to \mathrm{End}(V_\zeta \oplus V_\eta) = \mathrm{End}(\mathbb{O}_q \oplus \mathbb{O}_q)$ *in (9.19) is an algebra isomorphism.*

It is worth noting that at $q = 1$ the map in Proposition 9.20 reduces to

$$\begin{aligned} &\Psi : \mathbb{O} \to \mathrm{End}(\mathbb{O} \oplus \mathbb{O}), \\ &\quad v \mapsto \Psi(v) : x + y \mapsto v \cdot \overline{y} + \overline{x} \cdot v, \end{aligned} \tag{9.21}$$

so

$$\Psi(v)\Psi(v) : x + y \mapsto \Psi(v)(v \cdot \overline{y} + \overline{x} \cdot v) = v \cdot (\overline{v} \cdot x) + (y \cdot \overline{v}) \cdot v = (v|v)_1 (x + y)$$

(see Props. 2.21, 4.3). This shows that Ψ extends to the standard Clifford algebra—that is, to the quotient $T(\mathbb{O})/\langle v \otimes v - (v|v)_1 1\rangle$ of the tensor algebra, in this case as in [KPS].

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